A center is one of the borderline cases not marked on the main diagram of Fig. 6.7. It lies on the axis above zero. Physically it corresponds to an undamped spring that oscillates freely. That is a very special case, but in the applications it is extremely important. It represents *periodic motion*, and it is always present when $A^T = -A$—since skew-symmetric matrices have pure imaginary eigenvalues, and that is the requirement for a center.

Another marginal case lies on the parabola where the eigenvalues are equal. The two possibilities are in the figure below and in the exercises. If A is a multiple of the identity we have a ***star***; otherwise it is an ***improper node***.

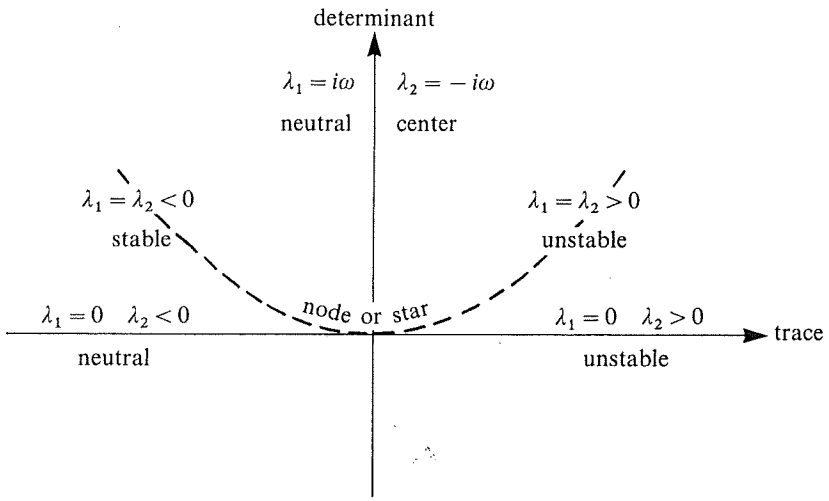


Fig. 6.10. Stability types at the region boundaries.

Stability for Nonlinear Systems

The equations in this section will not be linear; they will be linearized. They start in the form $u' = F(u)$. We continue to work with two equations in two unknowns, and a typical example is

$$\begin{aligned} u_1' &= au_1 - bu_1u_2 \\ u_2' &= cu_1u_2 - du_2. \end{aligned} \tag{3}$$

That is an important equation in population dynamics, where u_1 represents the population of the ***prey*** and u_2 is the population of the ***predator***. Because of the quadratic term u_1u_2, which is bad for the prey and good for the predator, the equation is nonlinear and an exact solution is not to be expected.

Our goal is not quantitative but qualitative—to get a clear idea of $u(t)$. All constants a,b,c,d are positive, and so are the initial conditions. Left alone, the prey

would increase ($u_1' = au_1$) and the predators would decrease ($u_2' = -du_2$). The number of meetings between them is proportional to u_1u_2, and each meeting gives the predator a chance to recoup. The model is similar to $u' = au - bu^2$ of Section 6.1, which had only one population—competing with itself. Here we have two populations, competing with each other according to $u_1' = F_1$ and $u_2' = F_2$.

The approach to linearized stability is straightforward:

1. Look for ***critical points*** u^* where $F(u^*) = 0$
2. Compute the matrix $A = \begin{bmatrix} \partial F_1/\partial u_1 & \partial F_1/\partial u_2 \\ \partial F_2/\partial u_1 & \partial F_2/\partial u_2 \end{bmatrix}$ at $u = u^*$
3. Decide the stability or instability of A from its eigenvalues.

A critical point is also a *stagnation point*; since $u' = F(u^*) = 0$ the solution does not move. It rests at equilibrium. The constant $u = u^*$ satisfies the differential equation and the orbit of that solution is a single point. The question is whether nearby solutions come in toward u^* (stability), circle around it (neutral stability), or leave it (instability). Except for the neutral case, which is always more delicate, that can be decided by linearizing the equation around u^*. We are looking at the equation locally, through a microscope that keeps the linear terms in $u - u^*$ and throws away quadratic and higher terms. If u leaves u^* we cannot tell where it goes.

The predator-prey example has two points where $F(u^*) = 0$:

$$\text{if} \quad au_1 = bu_1u_2 \quad \text{and} \quad cu_1u_2 = du_2 \quad \text{then}$$

$$\text{either} \quad u_1 = u_2 = 0 \quad \text{or} \quad u_1 = \frac{d}{c}, u_2 = \frac{a}{b}.$$

The derivatives of the two components of F go into A:

$$A = \begin{bmatrix} a - bu_2 & -bu_1 \\ cu_2 & cu_1 - d \end{bmatrix}.$$

At the critical point $u_1^* = u_2^* = 0$, this matrix is

$$A = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}.$$

The eigenvalues a and $-d$ have opposite signs, so (0,0) is a saddle point. It is unstable. From a small start the prey increases because $a > 0$; we leave the origin.

At the other point $u_1^* = d/c$, $u_2^* = a/b$, the matrix is

$$A = \begin{bmatrix} 0 & -bd/c \\ ca/b & 0 \end{bmatrix}.$$

The trace is zero and the determinant is ad. This critical point is a ***center***. The eigenvalues are $+i\sqrt{ad}$ and $-i\sqrt{ad}$. If the problem had been genuinely linear, not linearized, its solutions would oscillate with $\cos\sqrt{ad}\,t$ and $\sin\sqrt{ad}\,t$.

Remark The oscillation is now around u^*, not around the origin. The expansion $F(u) = F(u^*) + A(u-u^*) + \cdots$ starts with $F(u^*) = 0$, so the equation can be written $(u-u^*)' = A(u-u^*) + \cdots$. Linearization throws away the three dots. You see why only critical points are tested, since at other points the leading term is some nonzero value $F(u^*)$ and the solution goes steadily along. It is like minimization; we are not interested unless the derivative is zero.

The center point $u_1^* = d/c$, $u_2^* = a/b$ is the key to the predator-prey equation. Even though nonlinear, the problem has periodic solutions. There is a cycle in which the prey grows, the predator takes over, and then they both drop back to their starting values. The solution repeats, and in the next pages we show how to draw this periodic orbit (Fig. 6.13a below) in the "phase plane." First we summarize the analysis of linearized stability:

6D If $F(u^*) = 0$ then for initial values near u^* the nonlinear equation $u' = F(u)$ imitates the linearized equation with matrix $A = \partial F/\partial u$:

if A is unstable at u^* then so is $u' = F(u)$

if A is stable at u^* then so is $u' = F(u)$

The nonlinear equation has spirals, nodes, and saddle points according to A. However if A has a center (neutral stability with imaginary eigenvalues) the stability of $u' = F(u)$ is undecided.

This linearization extends to n equations and an n by n matrix A. But the phase plane will be special to $n = 2$, where curves have no room to escape. If one curve spirals, so must all the curves inside it—and the analysis is much more complete. Only the case of a center is not settled by linearization.

EXAMPLE The damped pendulum $\theta'' + c\theta' + \sin\theta = 0$

The extra term $c\theta'$ removes energy. Of course nonlinearity comes from $\sin\theta$. Prior to linearizing we write this equation as a first-order system $u' = F(u)$:

Reducing a second-order equation $\theta'' = G(\theta,\theta')$ ***to a first-order system***: Let $u_1 = \theta$ be the first unknown and introduce the "velocity" $u_2 = \theta'$ as a second unknown. Then the first equation is $u_1' = u_2$ and the second is $u_2' = G(u_1,u_2)$.

For the equation of a damped pendulum, this produces

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\sin u_1 - cu_2. \end{aligned} \tag{4}$$

The right side is $F(u)$. Its critical points and their stability come from the linearization steps 1–3:

(1) We solve $F(u^*) = 0$. In the first component $u_2^* = 0$. Then $\sin u_1^* = 0$ and u_1^* can be any multiple of π. There is an infinite row of critical points $u^* = (0,0),(\pm\pi,0), (\pm 2\pi,0), \ldots$. At even multiples of π the pendulum hangs down and at odd multiples it is stationary at the top.

(2) The derivative matrix (the Jacobian matrix $\partial F/\partial u$) is

$$A = \begin{bmatrix} \partial F_1/\partial u_1 & \partial F_1/\partial u_2 \\ \partial F_2/\partial u_1 & \partial F_2/\partial u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos u_1 & -c \end{bmatrix}.$$

At $u_1^* = 0$ the cosine is 1. At $u_1^* = \pm\pi$ it is -1. The matrix has one form for even multiples of π and another for odd multiples:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$$

(3) The trace of both matrices is $-c$. This is the diagonal sum, equal to the sum $\lambda_1 + \lambda_2$ of the eigenvalues. We are on the left side of the stability diagram in Fig. 6.7 if the damping constant c is positive.

The second matrix has determinant -1, giving a saddle point. All stationary points with the pendulum at the top are unstable. The first matrix has determinant $+1$, and the pendulum hanging down is stable. The exact type of stability depends on $(\text{trace})^2 - 4(\text{det})$, which is $c^2 - 4$:

$c^2 > 4$: A stable *node*, below the parabola (overdamping)

$c^2 = 4$: A stable *improper node*, on the parabola

$c^2 < 4$: A stable *spiral*, above the parabola (underdamping)

$c = 0$: A neutrally stable *center* (no damping)

$c = 0$ gives the equation $\theta'' + \sin\theta = 0$ of the last section, with steady oscillations. The energy was

$$\tfrac{1}{2}(\theta')^2 + 1 - \cos\theta = E = \text{constant}.$$

Now we want to draw this curve (which is closed) and the curves with damping (which are not closed) in the plane of θ and θ'. That is the celebrated phase plane.

The Phase Plane

In the phase plane we lose track of time. Like the orbit of a planet, the path is drawn without recording the time when it passes each point. However we have much more than just a path in space; it is a path in *phase space*, where the velocity is another coordinate. For one planet in 3-dimensional space the phase space is 6-dimensional, for N planets it is $6N$-dimensional, and for a 2 by 2 system it is a plane. Note that we could recover the time, if necessary, when the velocity is part of the picture; for a more general $u_1 - u_2$ plane, time is really lost. Note also that the equation $u' = F(u)$ is *autonomous*; F does not depend directly on t and the critical points u^* do not change with time. Those are the stagnation points at which $F(u^*) = 0$, and linearization gives a local picture around each of them. The object of the phase plane is to achieve a global picture of the motion.

The picture is easy when the energy is constant. For the undamped pendulum it was $E = \frac{1}{2}(\theta')^2 + 1 - \cos\theta$, and the paths in the phase plane are the curves $E =$ constant (Fig. 6.11). If the solution starts on a curve it stays on that curve. You recognize the saddle points at odd multiples of π. Those are stationary but unstable, at the top of the pendulum where $u' = 0$ and $\cos\theta = -1$. The energy is exactly $E = 2$. Smaller energies give the closed curves around the centers. Those are the ordinary back and forth oscillations, around the neutral equilibrium at the bottom. In the $\theta - \theta'$ plane they are closed curves on which energy passes between kinetic and potential. A closed curve is a periodic motion.

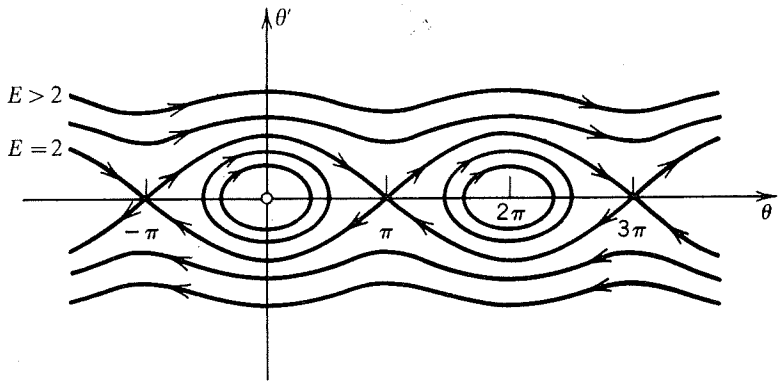


Fig. 6.11. The curves $E =$ constant for $\theta'' + \sin\theta = 0$: no damping.

For energies $E > 2$ the pendulum goes over the top. The curves continue on to $\pm 2\pi$ and never return to $\theta = 0$. Those are the upper and lower curves in Fig. 6.11, on which the velocity θ' is never zero. In that case the largest potential energy is 2 and some kinetic energy always remains. Note that where θ' is positive, above the horizontal axis, the arrows point to increasing θ.

Phase plane for the damped pendulum: $\theta'' + c\theta' + \sin\theta = 0$

If we multiply by θ' two terms can still be integrated but not the third:

$$\frac{d}{dt}\left[\tfrac{1}{2}(\theta')^2 + 1 - \cos\theta\right] = -c(\theta')^2. \tag{5}$$

The energy is no longer constant. In fact ***the energy is decreasing since the right side is negative.*** This changes the picture in the $\theta - \theta'$ plane to curves that spiral in to equilibrium.

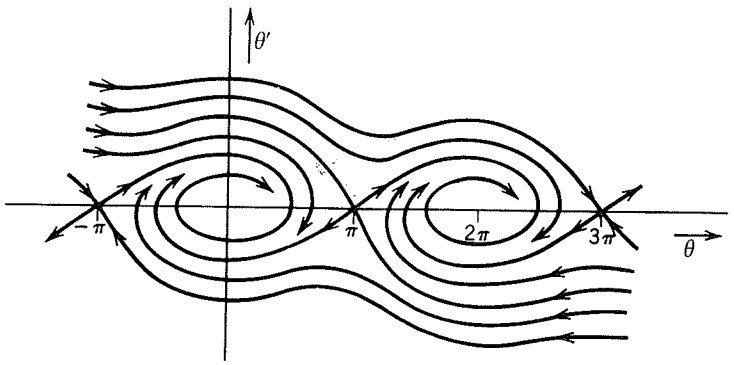


Fig. 6.12. Spiral points and stable equilibria: damped pendulum.

The initial conditions pick out a point on this plane—a starting value of θ and θ'. As t increases, the points $\theta(t)$, $\theta'(t)$ trace out a curve in the figure. When the initial condition is near equilibrium, the curve spirals immediately toward a focal point. In the physical plane, the pendulum goes back and forth but with less and less energy. That is the normal behavior. But if the initial velocity is large enough, then even with damping the pendulum makes it over the top. It is on one of the upper curves which reach $\theta = 2\pi$ or even $\theta = 4\pi$ before spiralling to equilibrium. The lower curves do the same at a point $\theta = -2n\pi$, when their energy is exhausted. There are no centers.

When the damping is greater, $c^2 \geq 4$, the picture changes again and the stationary points are *nodes*. Instead of going back and forth the pendulum swings back no more than once (Exercise 6.2.10). The spirals disappear and the curves go directly to equilibrium.

We want to discuss the "phase portraits" of three other problems. The first two can be drawn, but not the last. It comes from one of the basic unsolved problems of differential equations; please don't miss it.

Periods, Limit Cycles, and Strange Attractors

1. ***The predator-prey problem.*** The two equations were

$$u_1' = au_1 - bu_1u_2 \quad \text{and} \quad u_2' = cu_1u_2 - du_2.$$

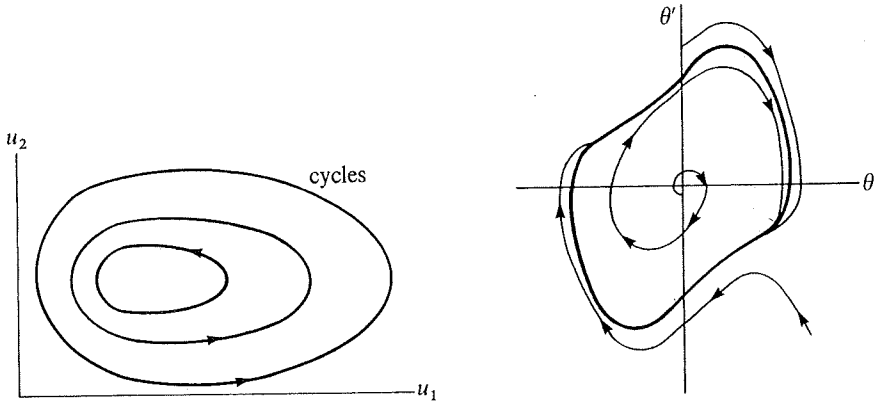
The first step is to find a single equation in the $u_1 - u_2$ plane. Roughly speaking, time is eliminated by using the chain rule:

$$\frac{du_2}{du_1} = \frac{du_2}{dt}\frac{dt}{du_1} = \frac{du_2/dt}{du_1/dt} = \frac{cu_1u_2 - du_2}{au_1 - bu_1u_2}. \tag{6}$$

This equation tells *where* the unknowns go but not *when*. It describes the orbits in the phase plane. At $u_1 = d/c$ the numerator is zero, the slope is zero, and the curve is horizontal. At $u_2 = a/b$ the denominator is zero, the slope is infinite, and the curve is vertical. In between we can solve equation (6) by separating u_1 from u_2:

$$\frac{du_2}{du_1} = \frac{(cu_1 - d)u_2}{(a - bu_2)u_1} \quad \text{or} \quad du_2\left(\frac{a}{u_2} - b\right) = du_1\left(c - \frac{d}{u_1}\right)$$

Integration gives $a \log u_2 - bu_2 = cu_1 - d \log u_1 + \text{constant}$. That takes some effort to plot carefully, but it is roughly an ellipse (Fig. 6.13a). The curve is closed and the predator-prey populations are *cyclic*. The initial values give a point in the $u_1 - u_2$ plane and the populations follow the curve from that point. In this model they return to their initial values and start again—a phenomenon that is not easy to see in experiment.



predator u_1 – prey u_2 convergence to limit cycle

Fig. 6.13. Phase planes for predator-prey and van der Pol ($c = 1$).

2. ***van der Pol's equation*** $\theta'' + c(\theta^2 - 1)\theta' + \theta = 0$. This describes damped oscillations, but it is not certain that the damping is positive. That changes with the

factor $\theta^2 - 1$. For small θ the damping is negative and the amplitude grows. For large θ the damping is positive and the solution decays. This may sound periodic but it is not. When the solution leaves a very small or very large value, it has no way to get back. Instead all orbits spiral toward a ***limit cycle*** which is the unique periodic solution to van der Pol's equation.

This illustrates how qualitative information comes from the phase plane:

> If there is a region R without critical points and a solution whose orbit stays in R, then either it is periodic or it spirals toward a closed curve (periodic solution) in the phase plane.

That is the Poincaré-Bendixson theorem, only possible for a plane. Van der Pol's equation cannot be solved exactly for $u = (\theta,\theta')$, whether time is left in or removed:

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= c(1-u_1^2)u_2 - u_1 \end{aligned} \quad \text{or} \quad \frac{du_2}{du_1} = \frac{c(1-u_1^2)u_2 - u_1}{u_2}. \tag{7}$$

However we can check that there are no critical points away from (0,0) and that orbits stay in a ring R. Therefore they approach a limit cycle.

3. ***Erratic oscillations from steady forcing.*** Suppose we combine two things that have been studied separately: nonlinear oscillations and a periodic force $f(t)$. Remember that linear oscillations had a natural frequency, and the force had a driving frequency. Nonlinear oscillations are different. The natural frequency depends on the amplitude, which is constantly changed by the force. For example:

(1) You push a nonlinear swing: $\theta'' + \sin\theta = A\cos\omega t$

(2) You bounce an elastic basketball by moving your hand up and down at a fixed frequency (it doesn't work too well)

(3) You send a small satellite into the periodic force field between twin stars.

In reality a satellite enters the solar system, but the N-body problem is unsolved and twin stars are easier. We know that two bodies in the x–y plane oscillate periodically around their center of mass. If they are the same size and a satellite arrives on the z-axis, it will stay on that axis—acted on symmetrically (and periodically) by gravity. Here is an amazing fact about its motion:

> If you prescribe any sequence of years, there is a satellite that crosses the x–y plane in exactly those years.

If the sequence is finite, the satellite escapes afterwards to infinity. If the crossings grow further apart, its oscillations are unbounded. If the years are equally spaced, the satellite can be periodic. This is all true when the orbits of the stars are near circles.

The nonlinear pendulum is similar; it can go over the top in any prescribed sequence of periods. The basketball has a crazy motion, especially when the hand