

PROVABILITY, SOUNDNESS AND COMPLETENESS

Rules of inference :

A mechanism for deriving new conclusions from premises.

e.g., MP :
$$\frac{\varphi, \varphi \Rightarrow \psi}{\psi}$$

Think (René), Think (René) \Rightarrow Exist (René)

$$\frac{\text{Think (René), Think (René)} \Rightarrow \text{Exist (René)}}{\text{Exist (René)}}$$

Other rules : Modus Tollens (MT), Modus Tollendo Ponens (MTP),
 double negation, And intro/elim, Or intro, ...
 DN AI AE OI

Soundness : $\frac{\varphi_1, \dots, \varphi_n}{\varphi}$ is sound iff all models of $\varphi_1, \dots, \varphi_n$ are models of φ (*)

e.g., $\frac{\neg\neg\varphi}{\varphi}$ is sound, $\frac{\neg(\varphi \Rightarrow \psi)}{\varphi \wedge \neg\psi}$ is sound (prove!)

(*) i.e., iff the premises entail the conclusion!

Equality & quantifier rules

$$S = : \frac{\tau = \tau', \varphi(\dots\tau\dots)}{\varphi(\dots\tau'\dots)}$$

sound

$$UI : \frac{(\forall v \varphi)}{\varphi[v/\tau]}$$

sound

" τ/τ' " = replace τ by τ'

$$EI : \frac{(\exists v \varphi)}{\varphi[v/\kappa]}$$

new

$$EG : \frac{\varphi(\dots\tau\dots)}{(\exists v \varphi(\dots v \dots))}$$

sound

e.g., $\frac{\text{Twain} = \text{Clemens}, \text{Clever}(\text{Twain})}{\text{Clever}(\text{Clemens})}$

$$\frac{\forall x. \text{Exists}(x)}{\text{Exists}(\text{René})}$$

← assuming, of course, that 'René' is in our FO language!

$$\frac{\exists x. \text{Place}(x) \wedge \text{Loc-at}(\text{Atlantis}, x)}{\text{Loc-at}(\text{Atlantis}, c)}$$

new constant

$$\frac{\text{Exists}(\text{René})}{\exists x. \text{Exists}(x)}$$

Soundness proofs for the substitution rules require induction on formula size & term size

Forward proof systems

Logical axioms (schemas) + forward inference rules

Axioms should be valid!

Some standard axioms

Assume we use only 2 connectives (\neg, \wedge) and one quantifier (\exists). Define the others as "shorthand", e.g., $(\varphi \Rightarrow \psi) \equiv_{df} \neg(\varphi \wedge \neg\psi)$

A1. $\varphi \Rightarrow (\psi \Rightarrow \varphi)$

A2. $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$

A3. $(\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\varphi \Rightarrow \psi)$

Q1. $(\forall v (\varphi \Rightarrow \psi)) \Rightarrow ((\forall v \varphi) \Rightarrow (\forall v \psi))$

Q2. $(\forall v \varphi) \Rightarrow \varphi[v/\tau]$ τ ground

Q3. $\varphi \Rightarrow (\forall v \varphi)$ v not free in φ

I1. $\tau = \tau$ τ ground

I2. $(\sigma = \tau) \Rightarrow (\varphi \Rightarrow \varphi_{\sigma/\tau})$

Rule of inference : MP $\frac{\varphi, \varphi \Rightarrow \psi}{\psi}$

"Easy" to prove things about, hard to prove things with!

Proofs

$\varphi_1, \varphi_2, \dots, \varphi_n$
 φ proof of φ from premises Δ

Each φ_i is

- in Δ , or
- an instance of a logical axiom
- the result of applying a rule of inference to a subset of $\{\varphi_1, \dots, \varphi_{i-1}\}$

non-logical axioms
 ("theory" of the domain)

Example

$\Delta = \{ \forall x. \exists y. \text{Loves}(x, y), \forall x. \text{Loves}(\text{John}, x) \Rightarrow x = \text{John} \}$

Prove: $\text{Loves}(\text{John}, \text{John})$

- | | |
|--|-----------------------|
| 1. $\forall x. \exists y. \text{Loves}(x, y)$ | Δ |
| 2. $\forall x. \text{Loves}(\text{John}, x) \Rightarrow x = \text{John}$ | Δ |
| 3. $\exists y. \text{Loves}(\text{John}, y)$ | UI, 1 |
| 4. $\text{Loves}(\text{John}, c)$ | EI (Skol), 3 |
| 5. $\text{Loves}(\text{John}, c) \Rightarrow c = \text{John}$ | UI, 2 |
| 6. $c = \text{John}$ | MP, 4, 5 |
| 7. $\text{Loves}(\text{John}, \text{John})$ | S ₅ , 6, 4 |

Provability (Derivability)

$\Delta \vdash \varphi$

$\{\} \vdash \varphi$ or $\vdash \varphi$ "theorem"

eg. axiom instances, $\neg\neg\varphi \Rightarrow \varphi$,
 $((\varphi \Rightarrow \psi) \Rightarrow (\neg\varphi \Rightarrow \neg\psi))$, ...

Δ inconsistent

$\Delta \vdash \square$

o.w. consistent

or \perp
 arbitrary contradiction,
 (or empty formula if
 we have $\varphi, \neg\varphi$)

Soundness & Completeness

Soundness

$\Delta \vdash \varphi$ implies that $\Delta \models \varphi$

Completeness

$\Delta \models \varphi$ implies that $\Delta \vdash \varphi$

Refutation completeness

If Δ is unsatisfiable, then $\Delta \vdash \square$