

Project: Ordinary Differential Equations (ODEs)

In MATLAB

4/19/2011

CSC 160

-----Abstract-----

The Ordinary Differential Equations Project is mainly about writing the derivative functions for our problem variants and using it in MATLAB's ODE solver (ode23). Through running the experiment, we will analyze “Balls in the Air”, “Attractors” problems. By running the solver to solve for the derivative functions, we can observe the original relationship between the variants and how the initial conditions affect the results.

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CSC 160
ODE Project
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Ordinary Differential Equations (ODEs) Project (MATLAB)

Introduction

The Ordinary Differential Equations (ODEs) Project is mainly about writing the derivative functions for our problem variants and using it in MATLAB's ODE solver (ode23). The derivative functions representing the relationship between the variants and their individual derivatives are derived from real world formula for physics or other scientific fields.

Through running the experiments in this project, we will analyze “Balls in the Air (Cannon at Crecy)”, “Attractors (Roessler Attractor, Lorenz Attractor, etc)” problems. By running the solver to solve for the derivative functions set, we can observe the original relationship between the variants and how the initial conditions affect the results. Especially, by plotting the results on a figure, we can also observe how each variant varies under different initial conditions.

Background for ODEs

(Brown C., Informal Introduction to Differential Equations)

Differential equations (DE's) can be used to express how related phenomena (variables) changing over time affect each other. That is, basically how everything in the world works.

In every case of ODE solution with MATLAB, the process can be roughly concluded in two steps:

1. Write a function that computes and returns the derivatives of the system's state vector.
2. Write a few lines to define a span of time over which you want solutions, the initial conditions of your problem, and invoke your chosen ODE solver (e.g. ode23()).

Balls in the Air (Trajectory of a projectile)

(en.wikipedia.org)

In physics, the ballistic trajectory of a projectile is the path that a thrown or launched projectile will take under the action of gravity, neglecting all other forces, such as friction from air resistance, without propulsion.

(Trajectory of a projectile with air resistance)

Air resistance will be taken to be in direct proportion to the velocity of the particle (i.e. $F_a \propto \vec{v}$). But this is valid only at low speed. At higher speed (for our case), the force of air resistance is proportional to the square of the particle's velocity (refer to the drag equation below).

Here, v_0 (or w), v_x (or u) and v_y (or v) will be used to denote the initial velocity below, the velocity along the direction of x and the velocity along the direction of y , respectively. The mass of the projectile will be denoted by m .

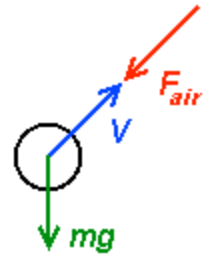


Figure 1.1 Force for Trajectory of a projectile with air resistance

Drag equation

(en.wikipedia.org)

The drag equation is a practical formula used to calculate the force of drag experienced by an object due to movement through a fully-enclosing fluid (or the air in our case).

The force on a moving object due to a fluid (or the air) is:

$$F_D = \frac{1}{2} \rho u^2 C_D A \quad \text{Eq. 1.0}$$

F_D is the drag force

ρ is the mass density of the fluid (or the air)

u is the velocity of the object relative to the fluid (or the air)

(denoted as u here for Eq. 1.0 only, will always be denoted as w)

A is the reference area

C_D is the drag coefficient (a dimensionless constant)

The reference area A is typically defined as the area of the orthographic projection of the object on a plane perpendicular to the direction of motion. For non-hollow objects with simple shape, such as a sphere, this is exactly the same as a cross sectional area. For other objects (for instance, a rolling tube or the body of a cyclist), A may be significantly larger than the area of any cross section along any plane perpendicular to the direction of motion.

(BlackBoard Assignment Page)

In addition, for drag coefficients, unstreamlined objects like cubes, flat surfaces normal to the airflow, and end-on cylinders tend to have drag coefficients pretty close to 1.0. Nice, streamlined, torpedo-shaped objects can have drag coefficients of 0.1 or even lower.

Attractors

(en.wikipedia.org)

An attractor is a set towards which a dynamical system evolves over time. That is, points that get close enough to the attractor remain close even if slightly disturbed.

Rössler Attractor

(en.wikipedia.org)

The Rössler attractor is the attractor for the Rössler system, a system of three non-linear ordinary differential equations. These differential equations define a continuous-time dynamical system that exhibits chaotic dynamics associated with the fractal properties of the attractor.

The original Rössler paper says the Rössler attractor was intended to behave similarly to the Lorenz attractor, but also be easier to analyze qualitatively. An orbit within the attractor follows an outward spiral close to the x,y plane around an unstable fixed point. Once the graph spirals out enough, a second fixed point influences the graph, causing a rise and twist in the z-dimension. In the time domain, it becomes apparent that although each variable is oscillating within a fixed range of values, the oscillations are chaotic. This attractor has some similarities to the Lorenz attractor, but is simpler and has only one manifold.

Rössler studied the chaotic attractor with $a = 0.2$, $b = 0.2$, and $c = 5.7$, though properties of $a = 0.1$, $b = 0.1$, and $c = 14$ have been more commonly used since.

Lorenz Attractor

(en.wikipedia.org)

The Lorenz attractor is an example of a non-linear dynamic system corresponding to the long-term behavior of the Lorenz oscillator. The Lorenz oscillator is a 3-dimensional dynamical system that exhibits chaotic flow, noted for its lemniscate shape. The map shows how the state of a dynamical system (the three variables of a three-dimensional system) evolves over time in a complex, non-repeating pattern.

Procedure (explanation of the functions used to experiment the methods)

"Balls in the Air" (Cannon at Crecy)

(BlackBoard)

Derived from Physics Laws:

Specified constant:

A is the cross-sectional area of the ball and m its mass ($A = \frac{\pi \cdot d^2}{4}$), ρ is the density of air (zero for a vacuum), g the force of gravity, and c is a dimensionless drag coefficient.

(Drag Coefficient)

For spheres, c tends to be about .5. Technically, c is the ratio between the actual drag force, and the theoretical drag force that would result from a hypothetical object that brought all the air it intercepted up to its own speed.

Unstreamlined objects like cubes, flat surfaces normal to the airflow, and end-on cylinders tend to have drag coefficients pretty close to 1.0. Nice, streamlined, torpedo-shaped objects can have drag coefficients of 0.1 or even lower.

Variants:

In particular, we want the force in each direction as a function of the x, y, and z velocity components ($x' = u$, $y' = v$ and $z' = w$).

Set $z = 0$ for the first four parts, the unit vector in the direction of instantaneous velocity is $(v_i \sin a, v_i \cos a)$ where $\sin a$ and $\cos a$ are the sine and cosine of the elevation angle of the direction of travel in the standard x-y system (x - horizontal, y - vertical).

Overall velocity: (summing up the velocity in each direction)

$$w^2 = u^2 + v^2 + z^2 \quad \text{Eq. 1.1}$$

From $F = ma$, or equivalently, $a = F/m$ we get

$$x'' = - \left(\frac{1}{2m} \right) (\rho c A w^2 \cos a) \quad (\text{drag alone}) \quad \text{Eq. 1.2}$$

$$y'' = -g - \left(\frac{1}{2m} \right) (\rho c A w^2 \sin a) \quad (\text{gravity plus drag}) \quad \text{Eq. 1.3}$$

Sines and cosines of the elevation angle a:

$$\cos a = \frac{u}{\sqrt{(u^2+v^2)}} \quad \text{Eq. 1.4}$$

$$\sin a = \frac{v}{\sqrt{(u^2+v^2)}} \quad \text{Eq. 1.5}$$

Indicated initial set up:

If we choose SI units (used to be called MKS), we can choose an iron cannonball of mass 5kg; then figure out its diameter from the density of iron: $7860 \frac{\text{kg}}{\text{m}^3}$.

Air density is about $1.23 \frac{\text{kg}}{\text{m}^3}$, and g is $\frac{9.81\text{m}}{\text{s}^2}$. For muzzle (initial) velocity, which should be subsonic for these equations, let's standardize on 150 m/sec, which is just under 1/2 the speed of sound.

Given the above and the chosen shooting elevation angle a, and you can calculate the initial velocities u_0 and v_0 .

Programming:

Then, from the above relations, we can write the derivative functions for our problem variants and use it in MATLAB's ODE solver (ode23). We will plot x (horizontal distance) against y (vertical height) to analyze the best elevation angle and longest range under different initial conditions.

In general, we will set up different initial values for each constant, experiment and compare the results for analysis. We will also run the solver over some angles in a range to find the best shooting angle and the longest distance.

(Continues on the next page)

Attractors:

Rössler Attractor:

Derivative functions:

$$\begin{aligned}x' &= -(y + z) \\y' &= x + ay \\z' &= b + xz - cz\end{aligned}$$

Lorenz Attractor

Derivative functions:

$$\begin{aligned}x' &= 10(y - x) \\y' &= -xz + \rho x - y \\z' &= xy - \frac{8z}{3}\end{aligned}$$

1. Experiment with different set of initial value set $y_0 = [x; y; z]$; compare the difference between the results. Compare the similarity between Rössler and Lorenz attractor.
2. Plot 3-D and 2-D figures to analyze the results.

Results (figures and statistical analysis)

& **Discussion** (including conclusions and the recommendations for future work)

"Balls in the Air" (Cannon at Crecy)

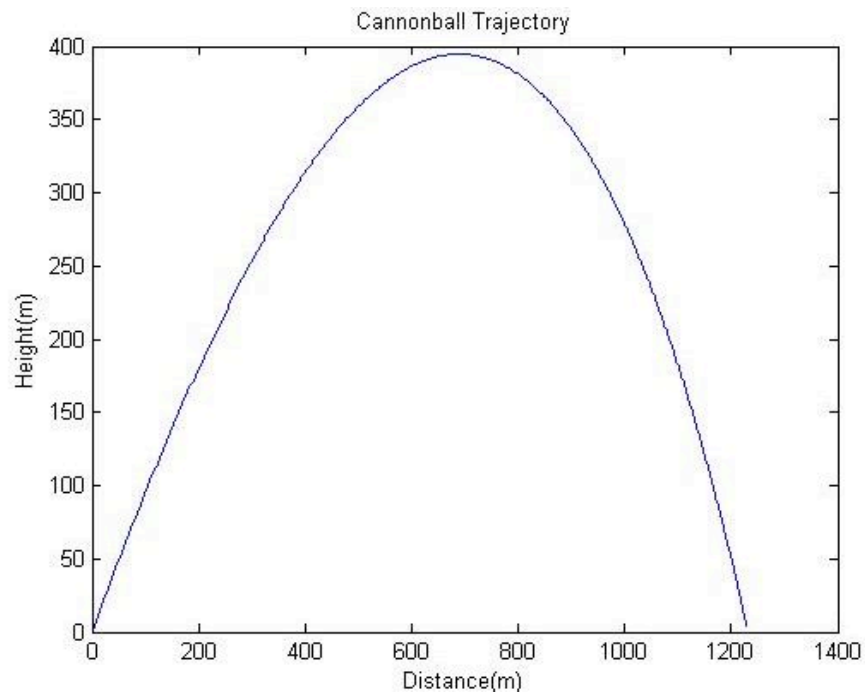


Figure 2.1 Cannon at Crecy_1_($m=5$, $c=.5$, $v_0=150$, angle = $\pi/4$)

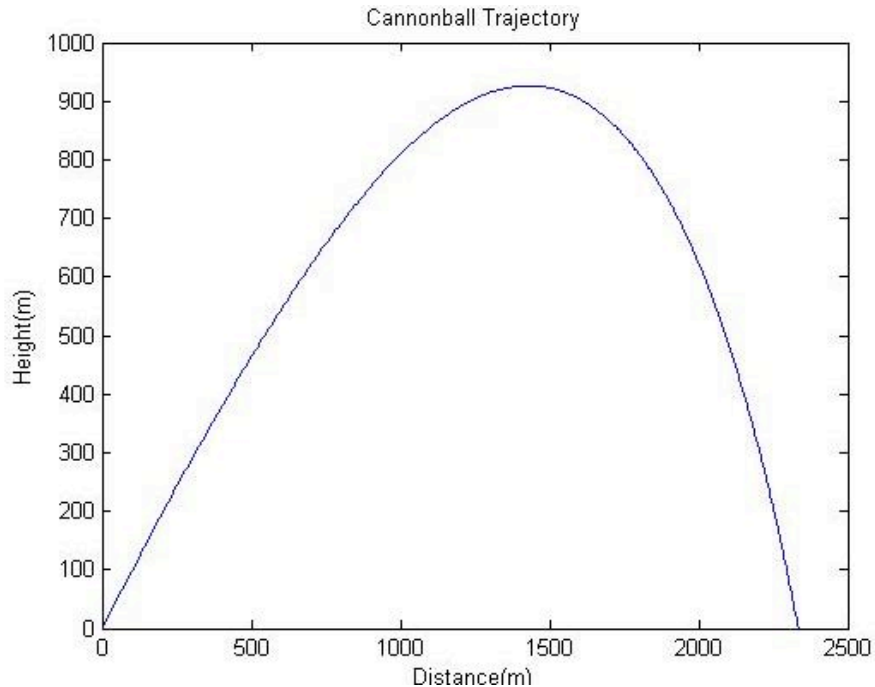


Figure 2.2 Cannon at Crecy_1_($m=5$, $c=.5$, $v_0=300$, angle = $\pi/4$)

Statistical Analysis:

From Figure 2.1 and 2.2, when launching angle = $\pi/4$, with 300m/s initial (/muzzle) velocity, the 5kg ball can reach about 2400 meters, longer than 1230 meters (approximate value) reached by $v_i = 150$ m/s. Also, with higher velocity, the trajectory curve which shows the path of the trajectory, will be more unbalanced (steeper on the right side when it starts to drop), which indicate the effect of air resistance.

Conclusion:

Thus, we conclude that with the same shooting angle, mass of the cannon ball (also same drag coefficient), the higher the initial velocity, the longer the horizontal distance can be reached, and the more the trajectory curve will be affected by the air resistance.

(from diary_shoot.m, by calling function 'range()' and 'max_range()', and print the result)

The longest distance possible with launch angle $\pi/4$ is 1231.650 m.
 The longest distance reached by launch angle from 0 to $\pi/2$ is 1244.3450 m.
 The best angle should be 0.2222 π .

For the simple canon ball trajectory considering the air resistance, the longest distance possible with launch angle $\pi/4$ is 1231.650 m. ($v_i = 150$ m/s)

However, the longest distance reached by launch angle from 0 to $\pi/2$ is 1244.3450 m and best angle should be 0.2222 π . ($v_i = 150$ m/s)

Thus, considering the effect of air resistance, the best shooting angle will not be $\pi/4$ (or 0.25 π), but will be a little smaller than that, 0.2222 π , approximately 40 degrees for elevation.

Meatball Trajectory

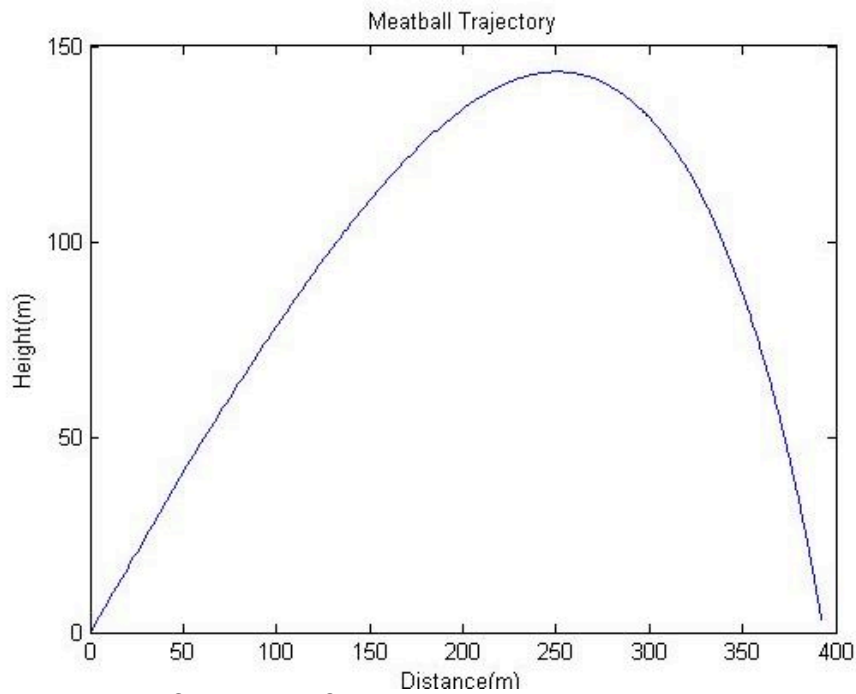


Figure 3.1 Cannon at Crecy_3_ ($m=0.02, c=.7, v_0=150, \text{angle}=0.2222\pi$)

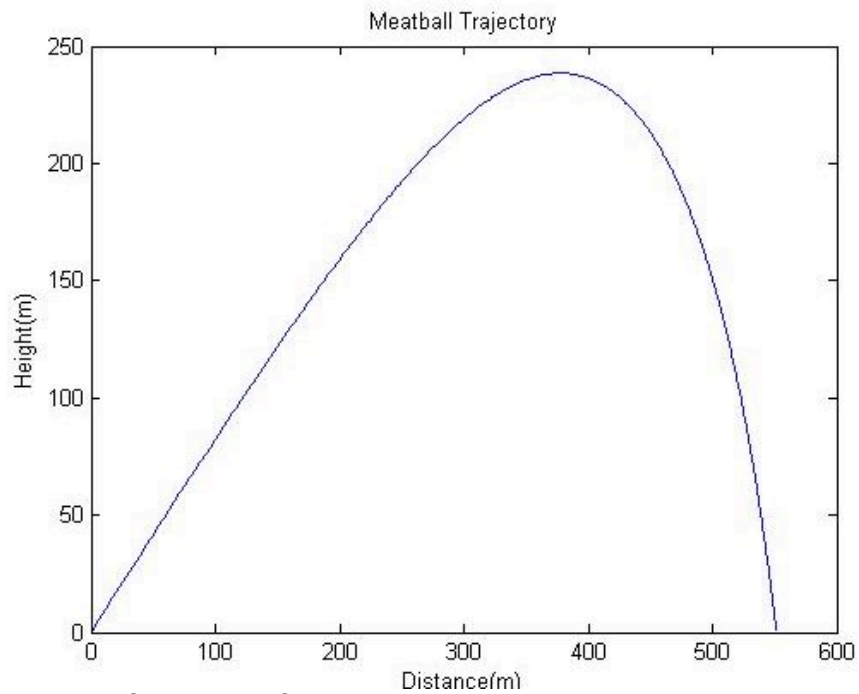


Figure 3.2 Cannon at Crecy_3_ ($m=0.02, c=.7, v_0=300, \text{angle}=0.2222\pi$)

Initial set change:

diameter meatball of the meatball = 15/100 m

drag coefficient = 0.7 (given value)

density of meat = $\frac{1042\text{kg}}{\text{m}^3}$ (given value)

Volume of the meatball = $\frac{4}{3} * \pi * \left(\frac{\text{diameter}}{2}\right)^3$ (calculated the volume by diameter)

mass = (Volume of the meatball)*(density of meat) (calculate the mass by volume times the density)

Statistical Analysis:

From Figure 3.1 and 3.2, when launching angle = 0.2222π , with 300m/s initial (/muzzle) velocity, the meatball can reach about 550 meters, longer than 380 meters (approximate value) reached by $v_i = 150\text{m/s}$. Also, with higher velocity, the trajectory curve which shows the path of the trajectory, will be more unbalanced (steeper on the right side when it starts to drop), which indicate the effect of air resistance.

Conclusion:

Thus, we conclude that with the same shooting angle, mass of the meatball (also same drag coefficient 0.7), the higher the initial velocity, the longer the horizontal distance can be reached, and the more the trajectory curve will be affected by the air resistance.

(from diary_shoot.m, by calling function 'range()' and 'max_range()', and print the result)

```
For meatball, the longst distance possible with launch angle 0.2222pi is
380.290 m.
(meatball)The longest distance reached by launch angle from 0 to pi/2 is
400.9924 m.
The best angle should be 0.1778pi.
```

For the simple meatball trajectory considering the air resistance, the longest distance possible with launch angle 0.2222π is 380.290 m. ($v_i = 150\text{m/s}$)

However, the longest distance reached by launch angle from 0 to $\pi/2$ is 1244.3450 m and best angle should be 0.2222π . ($v_i = 150\text{m/s}$)

Thus, considering the effect of air resistance, the **best shooting angle** will not be 0.2222π (or 0.25π), but will be smaller than that, 0.1778π , approximately 32 degrees for elevation. With lighter in weight and smaller in Area, larger in drag coefficient, the best shooting angle for the meatball will be smaller than that of the cannon ball.

Compare to the cannonball, the meatball is lighter, but it will fly shorter distance, also the maximum height reached by the meatball will be lower. From the curve on the figure, we might guess that the meatball will be affected by air resistance more.

Cannonball Trajectory (consider the variation of the density of air)

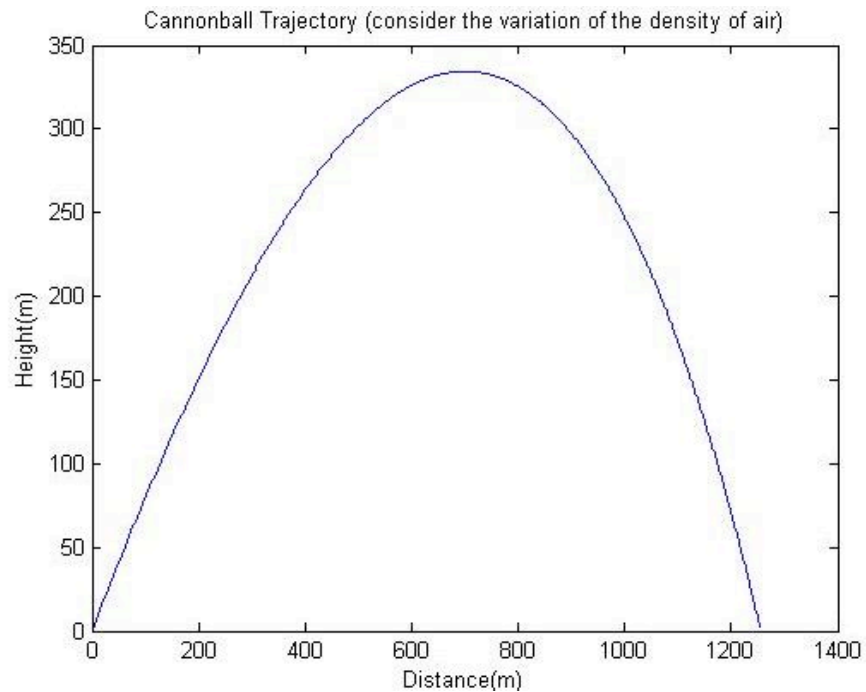


Figure 4.1 Cannon at Crecy_4 (air)_(m=5,c=.5,v₀=150, angle = 0.2222pi)

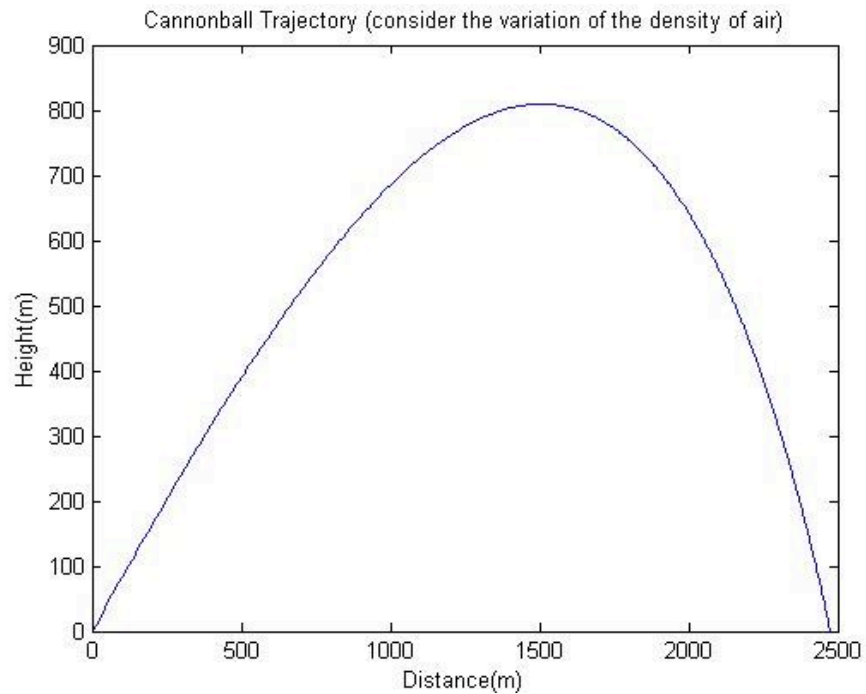


Figure 4.2 Cannon at Crecy_4 (air)_(m=5,c=.5,v₀=300, angle = 0.2222pi)

Initial set change:

Density of air need to be calculated, instead of the given fixed value

$$\text{density} = (p * M)/(R * T)$$

How to get each factor for calculating the air density:

sea level standard atmospheric pressure $p_0 = 101325$ Pa

sea level standard temperature $T_0 = 288.15$ K

temperature lapse rate $L = 0.0065$ K/m

universal gas constant $R = 8.31447$ J/(mol·K)

molar mass of dry air $M = 0.0289644$ kg/mol

Temperature at altitude h meters above sea level is given by:

$$T = T_0 - L * y(2) \quad \text{Eq. 2.1}$$

The pressure at altitude h is given by:

$$p = p_0 * \left(1 - L * \frac{y(2)}{T_0}\right)^{\frac{g * M}{R * L}} \quad \text{Eq. 2.2}$$

Statistical Analysis:

From Figure 4.1 and 4.2, when launching angle = 0.2222π , with 300m/s initial (/muzzle) velocity, the cannonball can reach almost 2500m, longer than 1250 meters (approximate value) reached by $v_i = 150$ m/s. However, with higher velocity, the trajectory curve which shows the path of the trajectory, will not be obviously unbalanced more (not obviously steeper on the right side when it starts to drop), which indicate the effect of air resistance is less when considering the variation of air density.

Conclusion:

Thus, we conclude that with the same shooting angle, mass of the cannon ball (also same drag coefficient 0.5), considering the variation of air density, the higher the initial velocity, the longer the horizontal distance can still be reached, but the trajectory curve will not necessarily be affected by the air resistance too much.

(from diary_shoot.m, by calling function 'range()' and 'max_range()', and print the result)

```
Considering the variation of the density of air,  
the longest distance possible with launch angle pi/4 is 2334.619 m.  
Considering the variation of the density of air,  
the longest distance reached by launch angle from 0 to pi/2 is 2486.6234 m.  
The best angle should be 0.2000pi.
```

For the cannon trajectory considering the air resistance and the variation of air density, the longest distance possible with launch angle $\pi/4$ is 2334.619 m. ($v_i = 300$ m/s)

However, the longest distance reached by launch angle from 0 to $\pi/2$ is 2486.6234m and best angle should be 0.2π . ($v_i = 300$ m/s)

Thus, considering the effect of air resistance and the variation of air density, the best shooting angle will not be 0.25π (or 0.2222π), but will be smaller than that, 0.2π , approximately 36 degrees for elevation.

Compare to the cannonball case without considering the variation of air density, the trajectory curve will be affected by the air resistance less.

Cannonball Trajectory (including the z-direction wind)

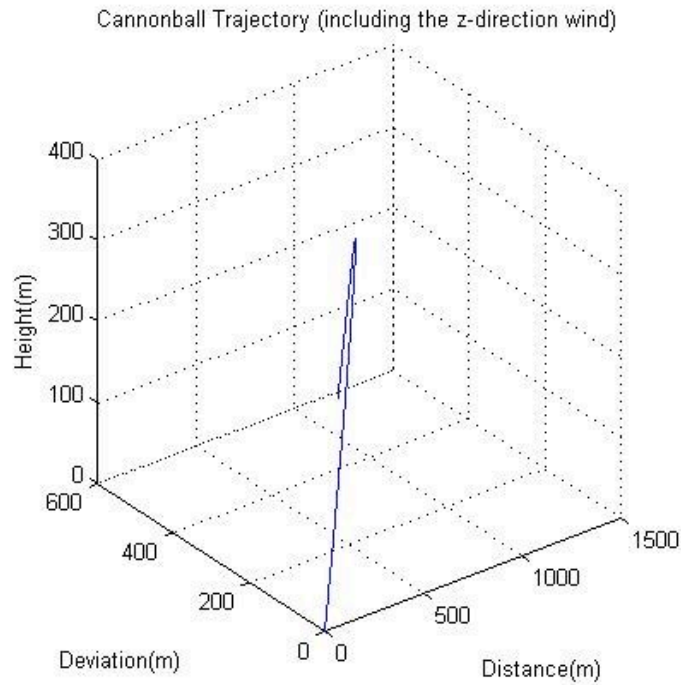


Figure 5.1 Cannon at Crecy_5_($m=5, c=.5, v_0=150$) - 3d

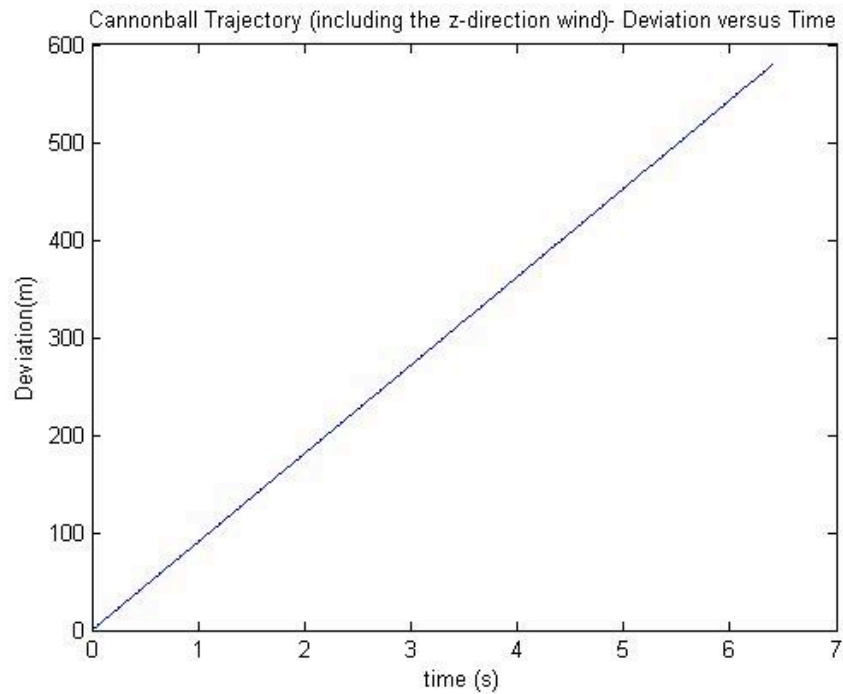


Figure 5.2 Cannon at Crecy_5_($m=5, c=.5, v_0=150$) - Deviation vs Time

Statistical Analysis & Conclusion:

From Figure 5.1 and 5.2, the z-direction wind (80mph Sirocco) will make the trajectory deviate about 600 meters, so to fix this, the initial shooting direction need to be adjusted.

From Figure 5.2, we can tell that since the distance deviated is proportional to the flying time, the longer time the cannon ball flies, the further it will deviate.

Attractors

Roessler Attractor

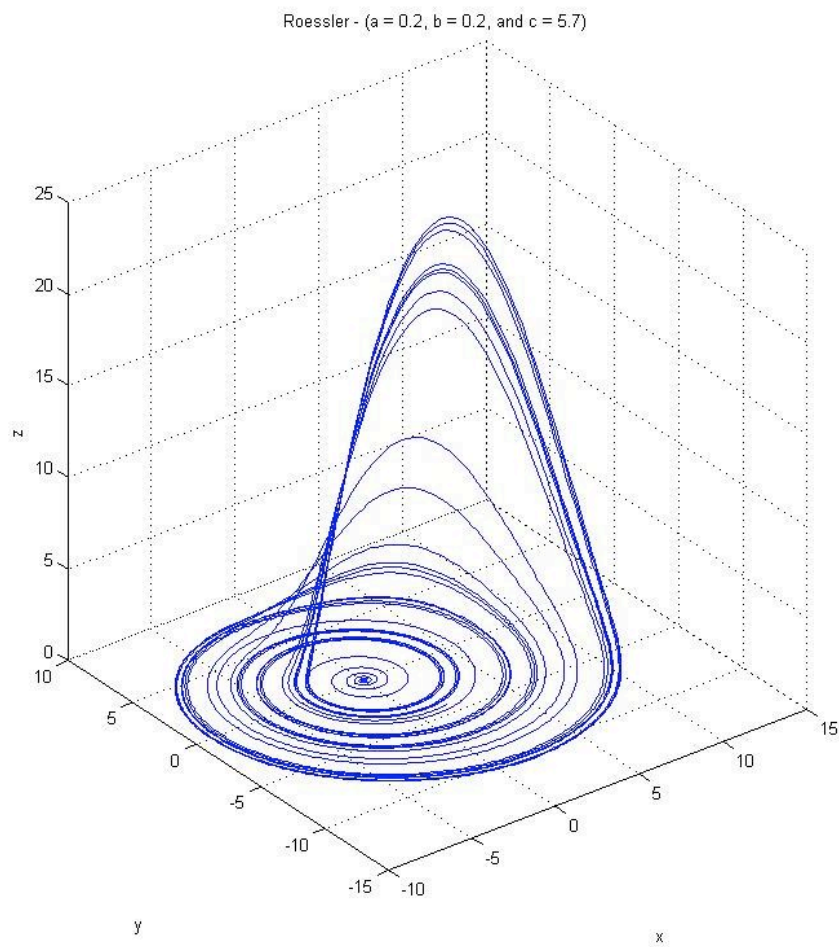


Figure 6.1(a) Roessler - smaller time span (a = 0.2, b = 0.2, and c = 5.7)

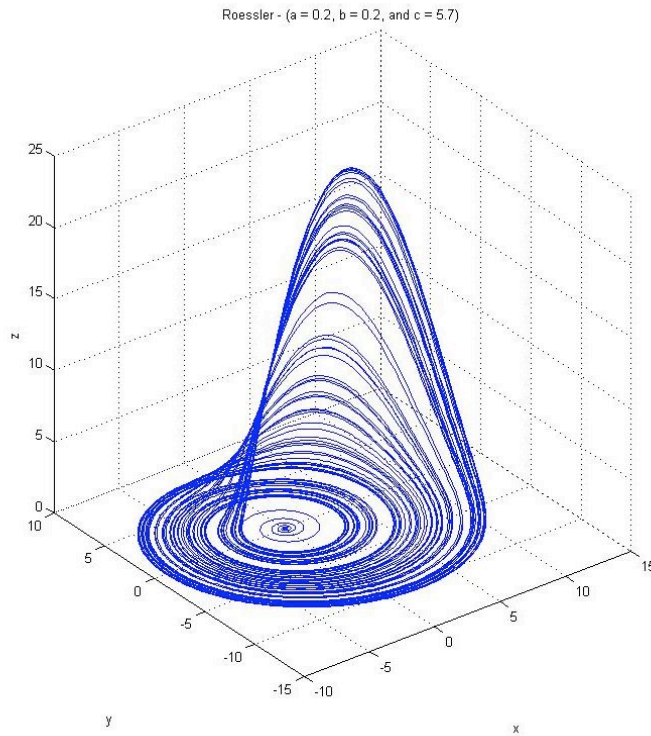


Figure 6.1(b) Roessler - larger time span ($a = 0.2$, $b = 0.2$, and $c = 5.7$)

Statistical Analysis & Conclusion:

From Figure 6.1 (a) and 6.1 (b), it shows what the data plots will look like for parameter values: $a = 0.2$, $b = 0.2$, and $c = 5.7$ (studied for the chaotic attractor)

Comparing the result from smaller time span and larger time span, we can observe that the maximum curve for each axis (the outside “range”) will not change with time span. Therefore the figure reached by the attractor does have a maximum point.

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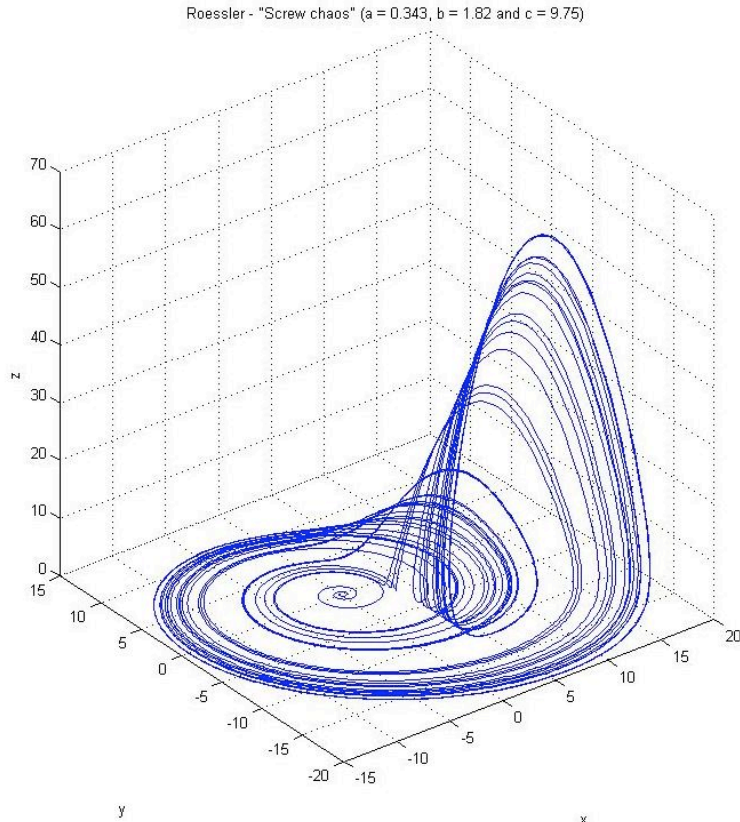


Figure 6.2(a) Roessler - smaller time span ($a = 0.343$, $b = 1.82$, and $c = 9.75$)

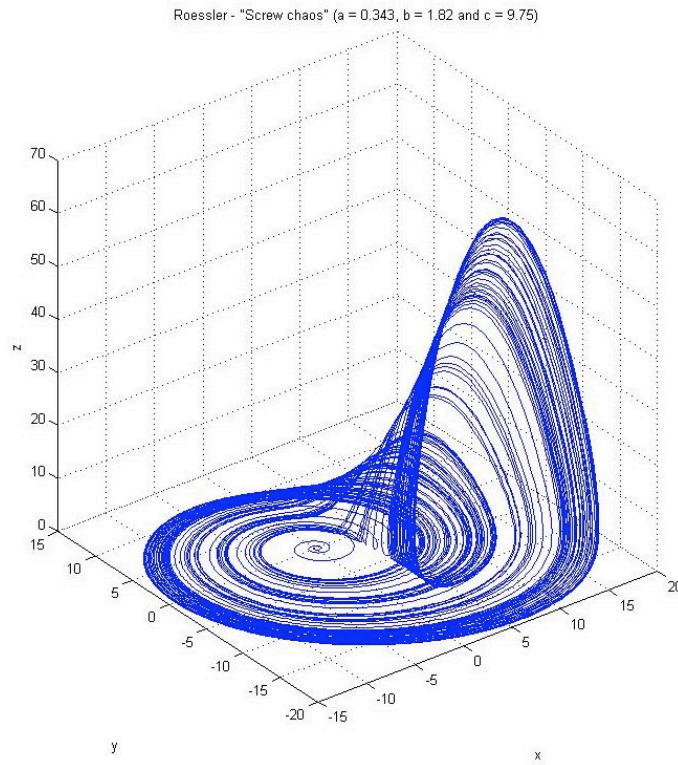


Figure 6.2(b) Roessler - larger time span ($a = 0.343$, $b = 1.82$, and $c = 9.75$)

Statistical Analysis & Conclusion:

From Figure 6.2 (a) and 6.2 (b), it shows what the data plots will look like for parameter values: $a = 0.343$, $b = 1.82$, and $c = 9.75$ (studied for the chaotic attractor)

Comparing the result from smaller time span and larger time span, we can still observe that the maximum curve for each axis (the outside “range”) will not change with time span. Therefore the figure reached by the attractor does have a maximum point.

Lorenz Attractor

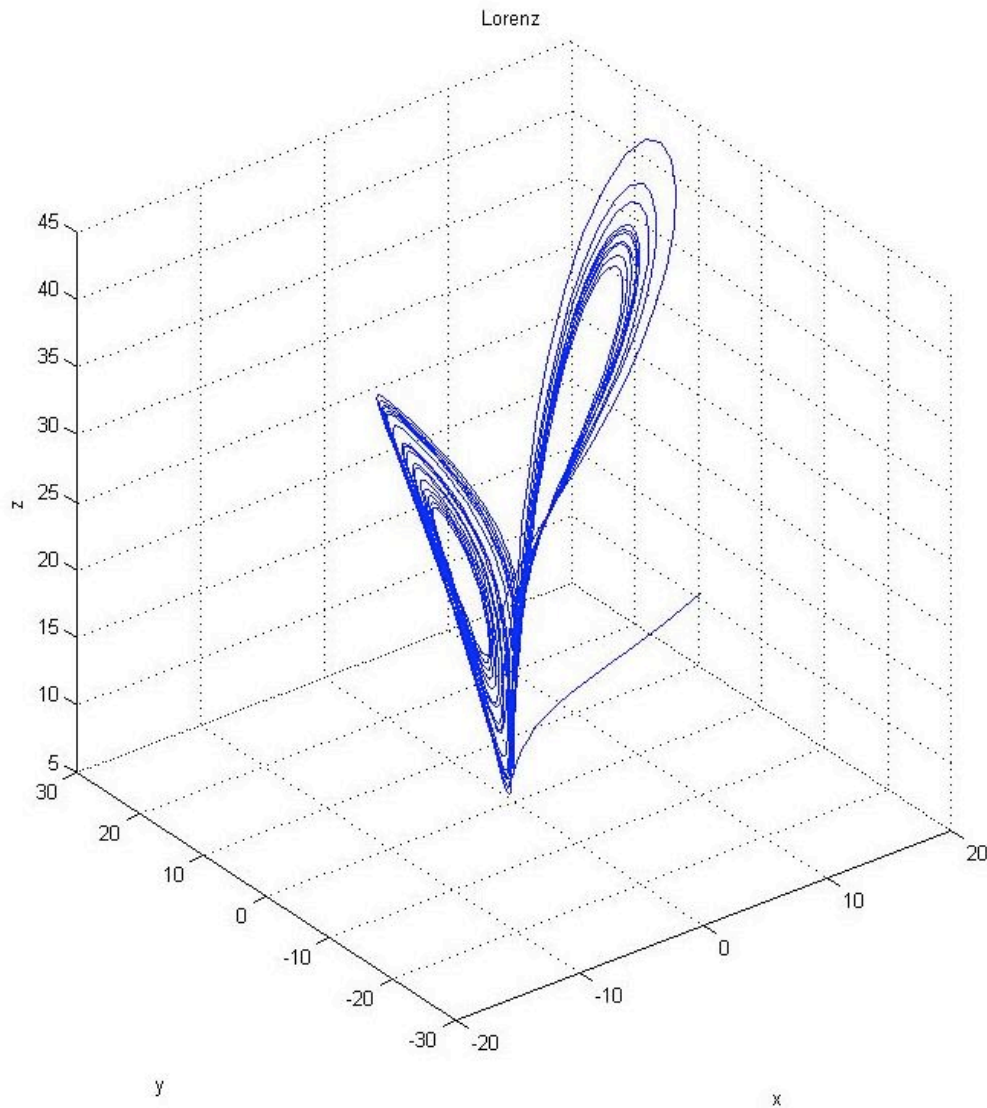


Figure 7.1(a) Lorenz ($\rho=28$)

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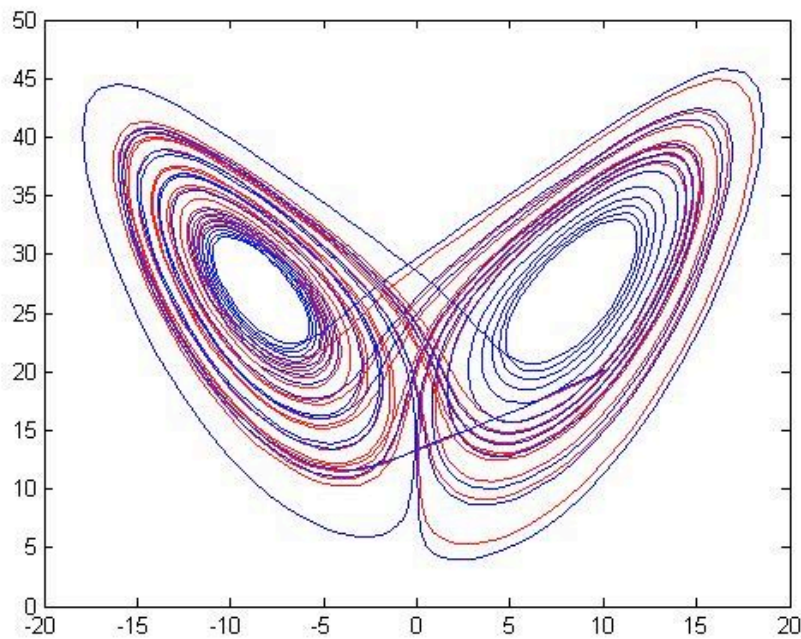


Figure 7.1(b) Lorenz – Difference (plotting x-z) ($\rho = 28$)

Statistical Analysis & Conclusion:

From Figure 7.1 (a) and 7.1 (b), it shows what the Lorenz attractor shares some similarities with the Roessler Attractor.

From Figure 7.1 (b), although the new initial conditions (results from which are plotted in blue curve) are close to the previous conditions (results from which are plotted in red curve), but there is big difference in the two results.

References

Brown, C. 2010 Informal Introduction to Differential Equations

The main ideas of the functions (including the inside computing method) mainly originated from BlackBoard CSC 160 ODEs assignment page.

Author: Chris Brown

Complete URL:

http://my.rochester.edu/webapps/portal/frameset.jsp?tab_tab_group_id= 2_1&url=/webapps/blackboard/execute/courseMain?course_id= 40459_1

Date that accessed the site: 04/19/2011 (last access)

Author: (source) Wikipedia.org

Complete URL: <http://en.wikipedia.org/wiki/Attractor>

Date that accessed the site: 04/19/2011 (last access)

Author: (source) Wikipedia.org

Complete URL: http://en.wikipedia.org/wiki/Drag_equation

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Author: (source) Wikipedia.org

Complete URL: http://en.wikipedia.org/wiki/Trajectory_of_a_projectile

Date that accessed the site: 04/19/2011 (last access)