

***Computing Dehn Twists and Geometric
Intersection Numbers in Polynomial
Time***

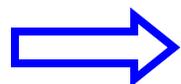
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DePaul University (Chicago)

Daniel Štefankovič

University of Rochester

outline



What is a curve?

Some history of algorithmic problems.

Representing surfaces.

Representing simple curves in surfaces.

Transforming between various representations.

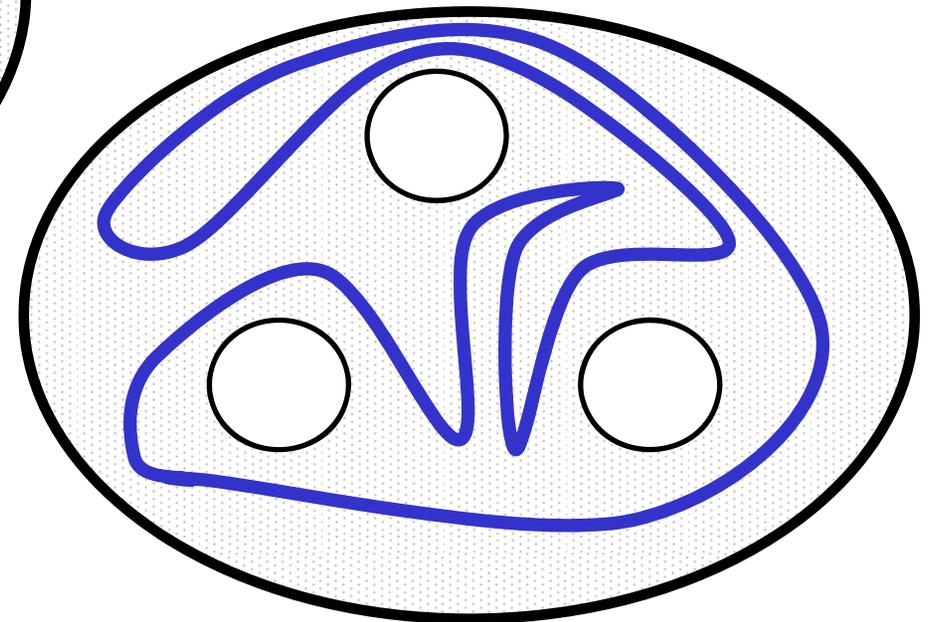
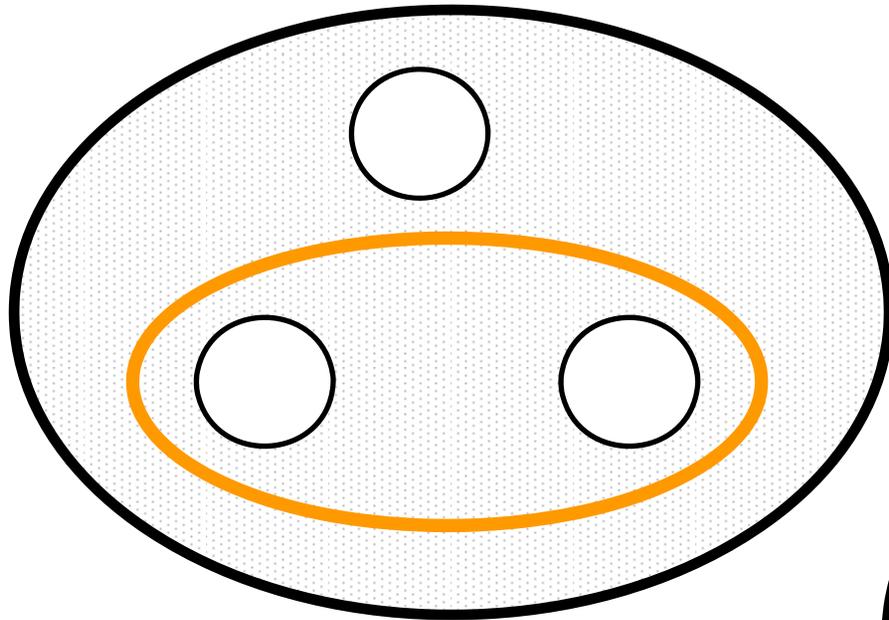
TOOL: Word equations.

What is a Dehn twist and why is it interesting?

Computing Dehn twists.

Open questions.

Curves on surfaces

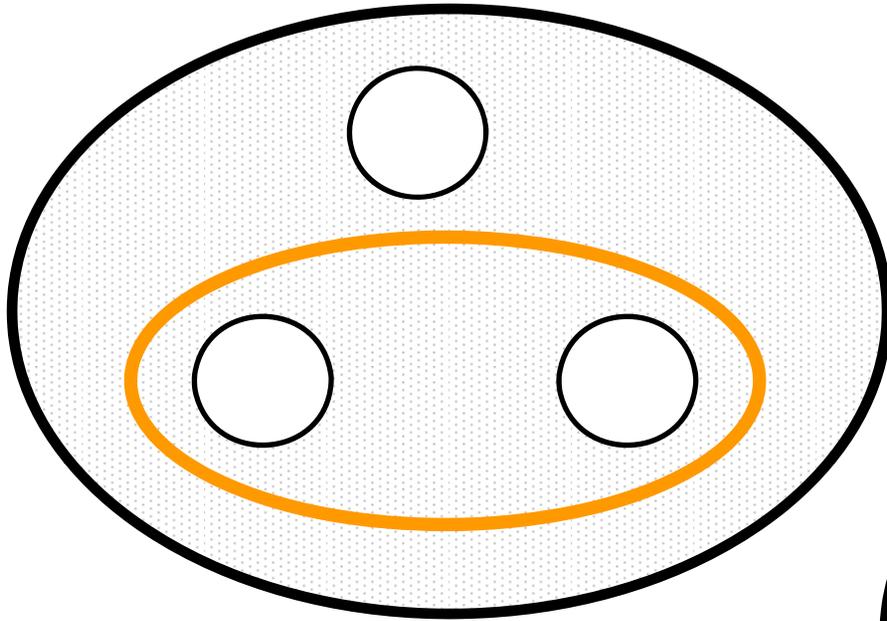


different? same?

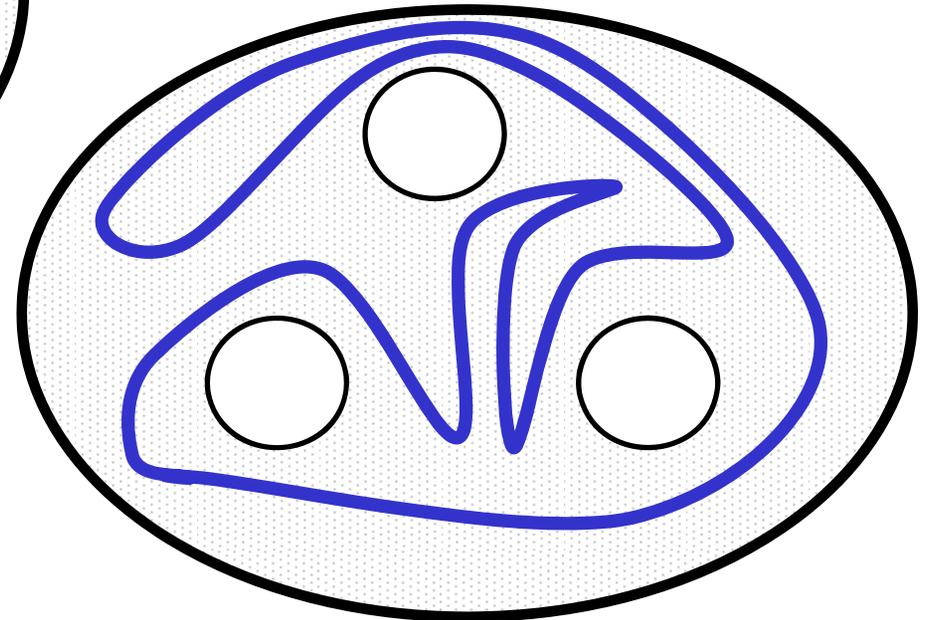
closed curve = homeomorphic image of circle S^1

simple closed curve = γ is injective (no self-intersections)

Curves on surfaces

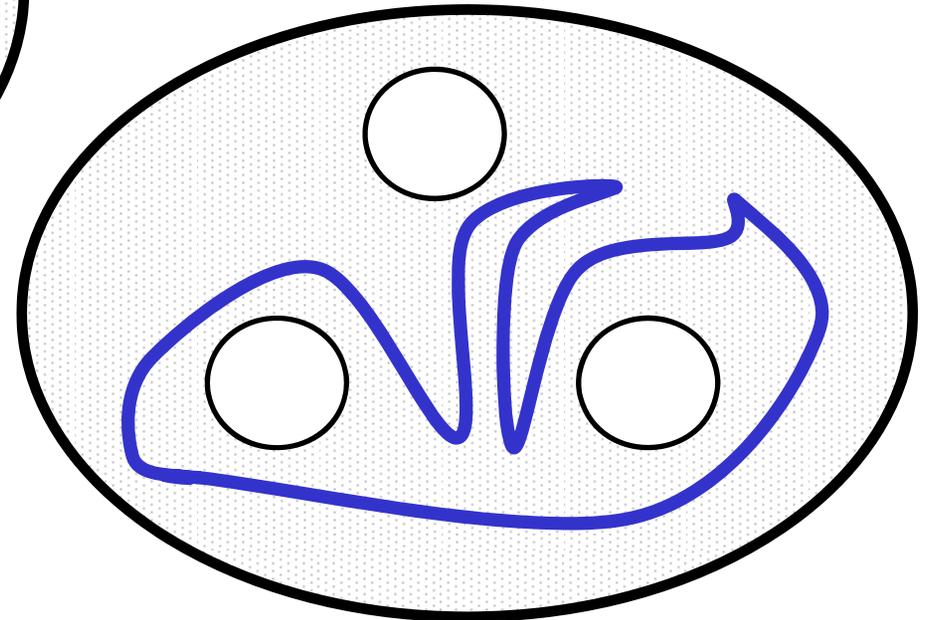
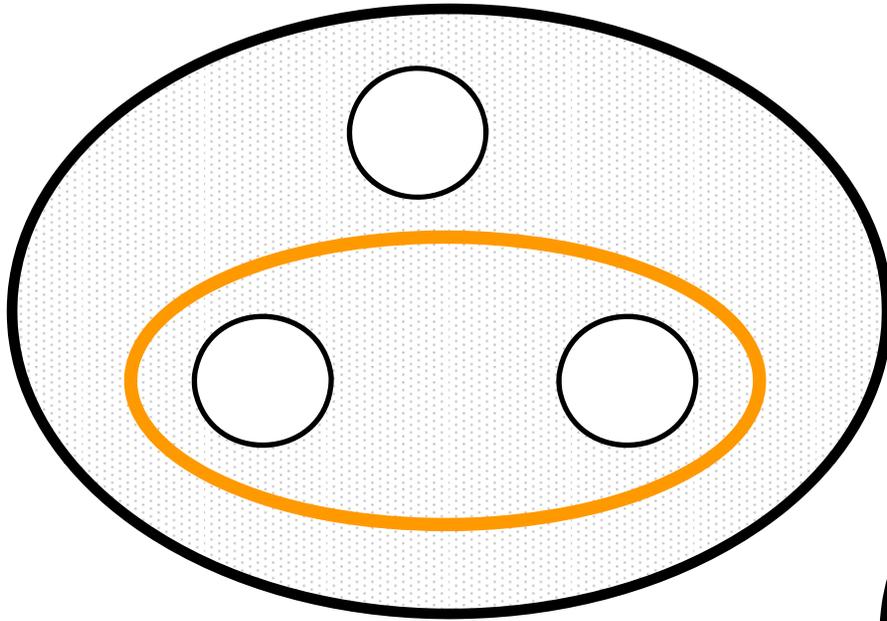


different? same?



homotopy equivalent

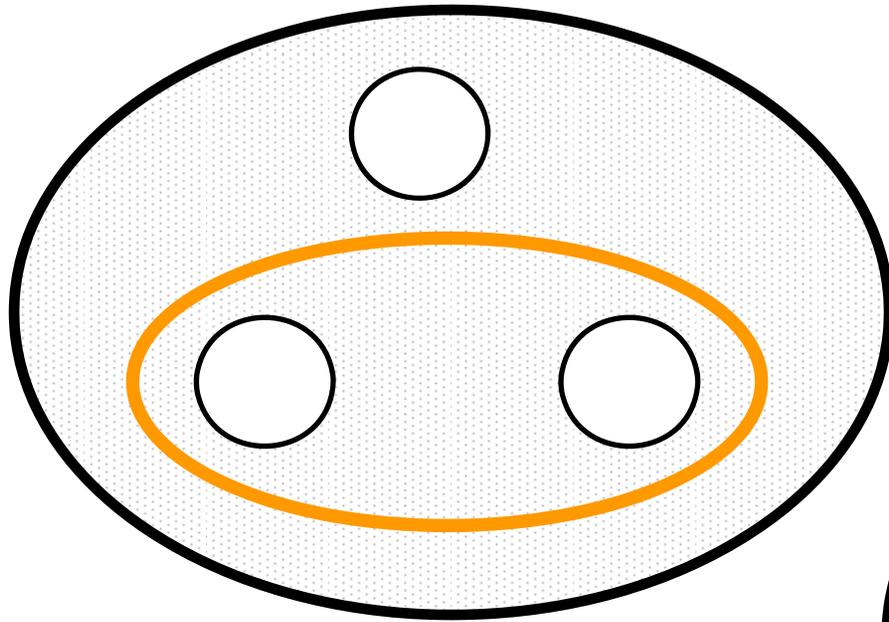
Curves on surfaces



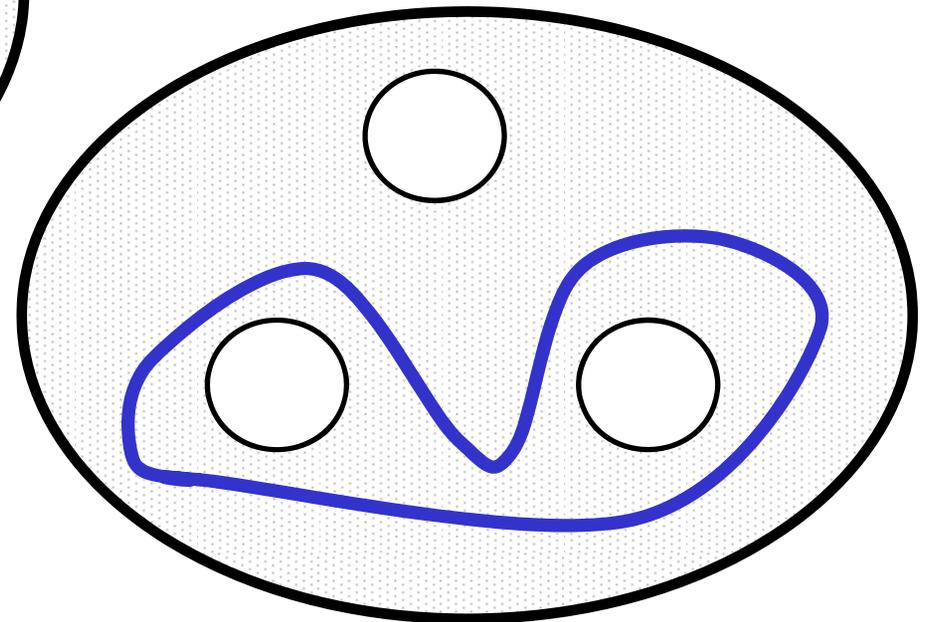
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Curves on surfaces

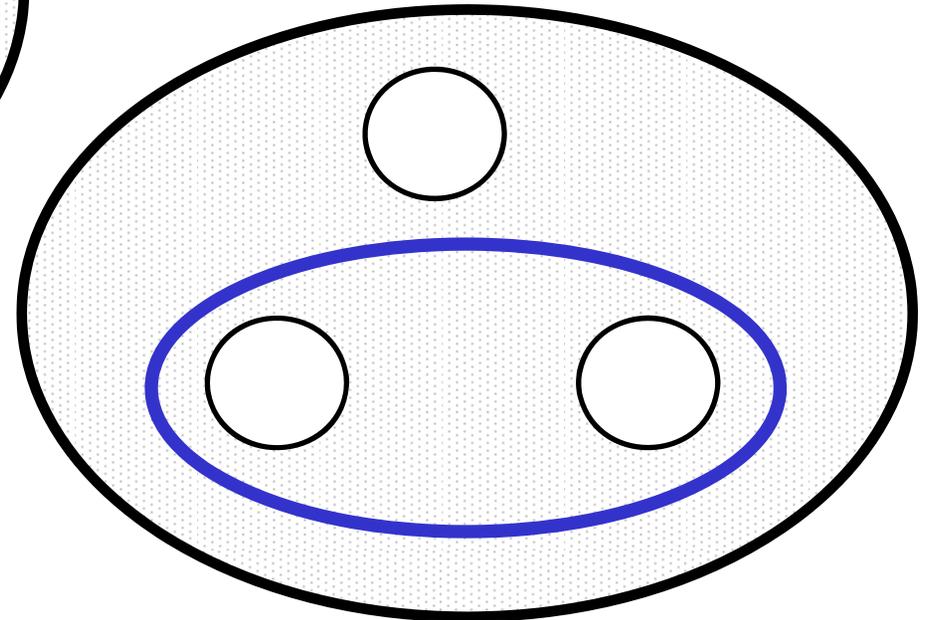
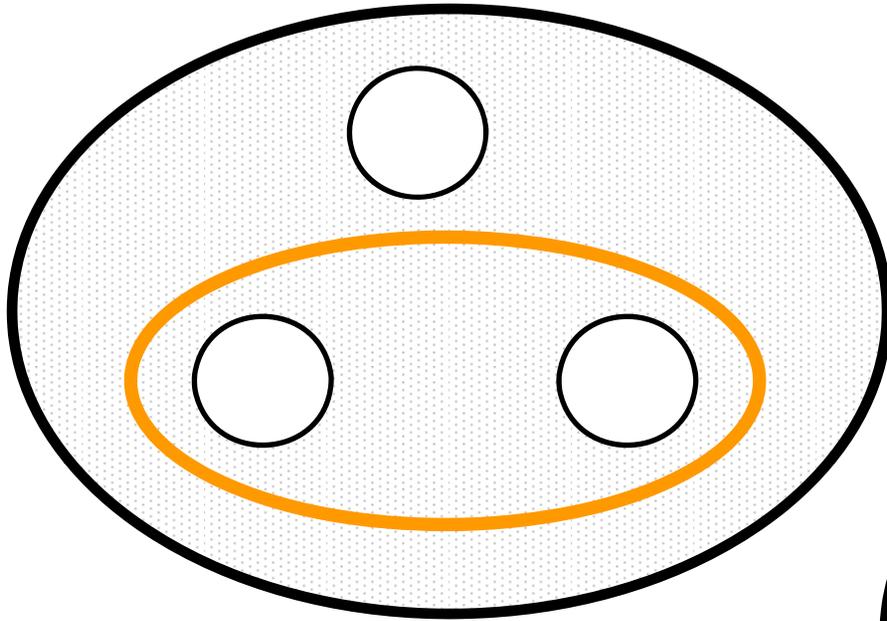


different? same?



homotopy equivalent

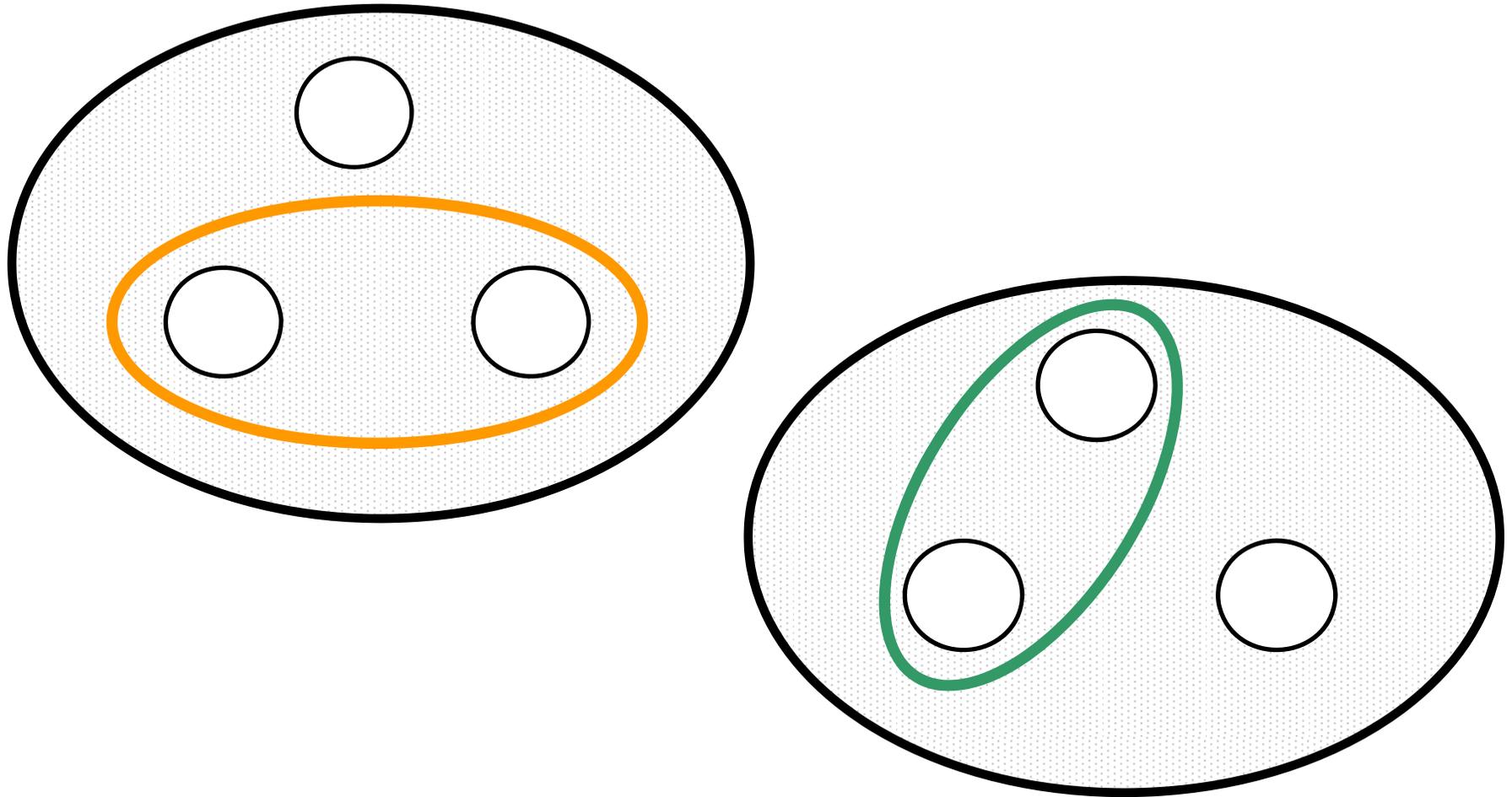
Curves on surfaces



different? same?

homotopy equivalent

Curves on surfaces



not homotopy equivalent

Curves on surfaces

curves

continuous objects

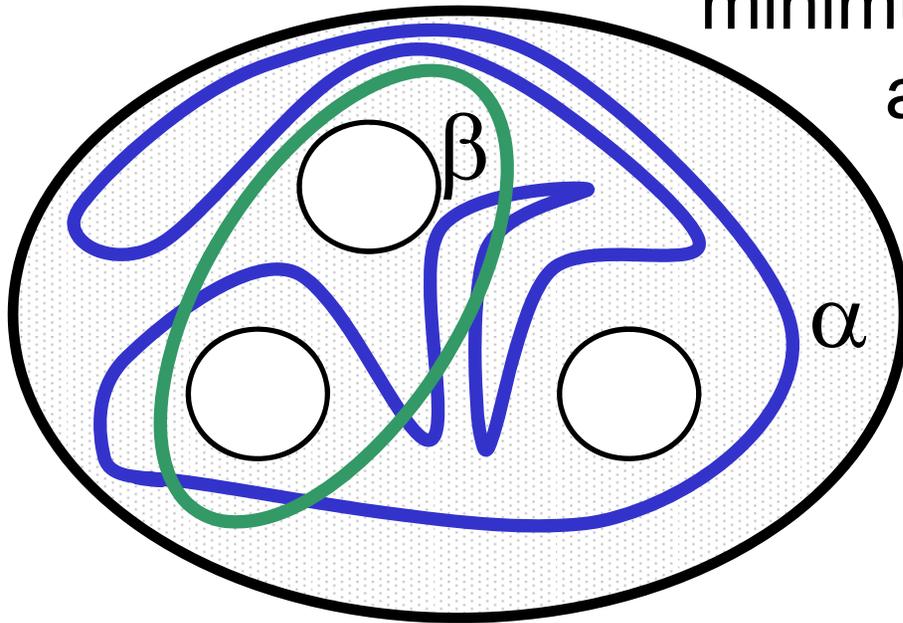
homotopy classes of curves

combinatorial objects

- 1) how to represent them?
- 2) what/how to compute?

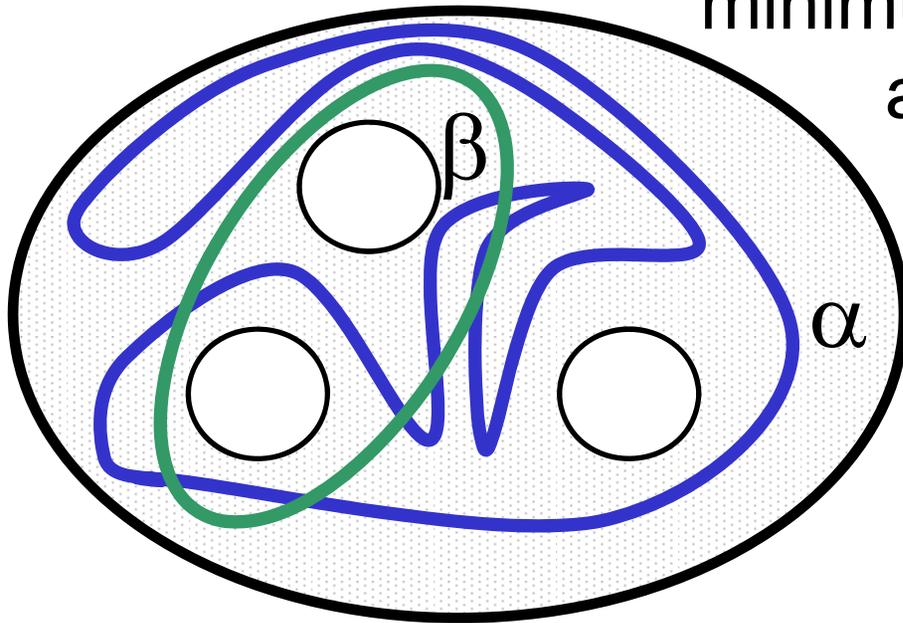
Geometric intersection number

minimum number of intersections
achievable by continuous
deformations.

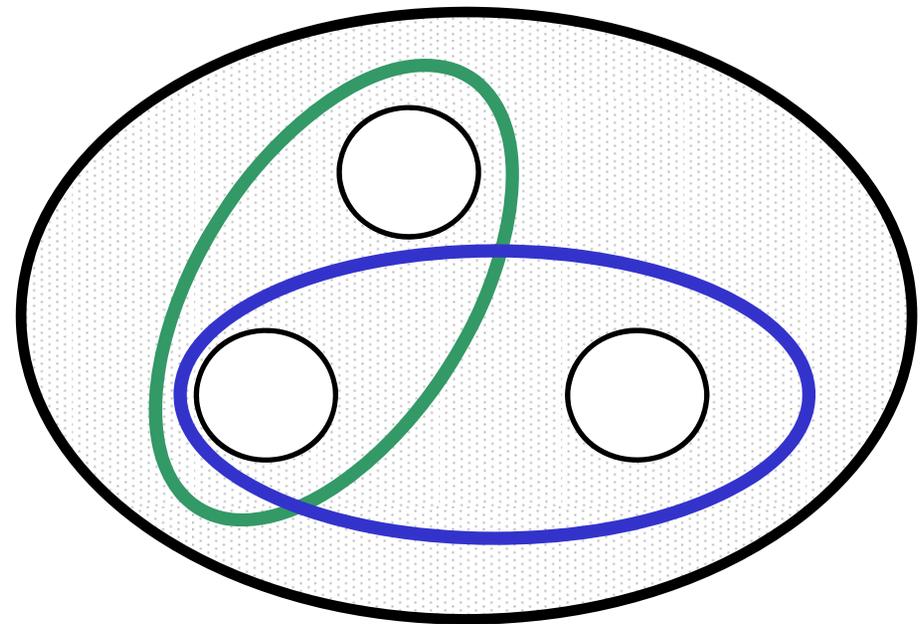


Geometric intersection number

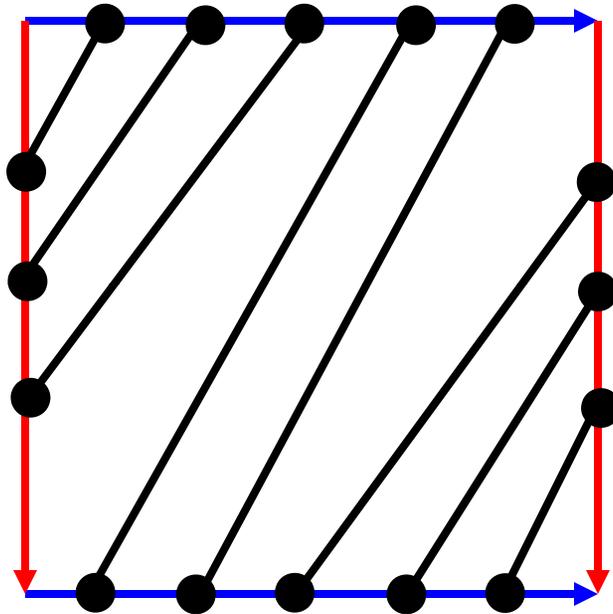
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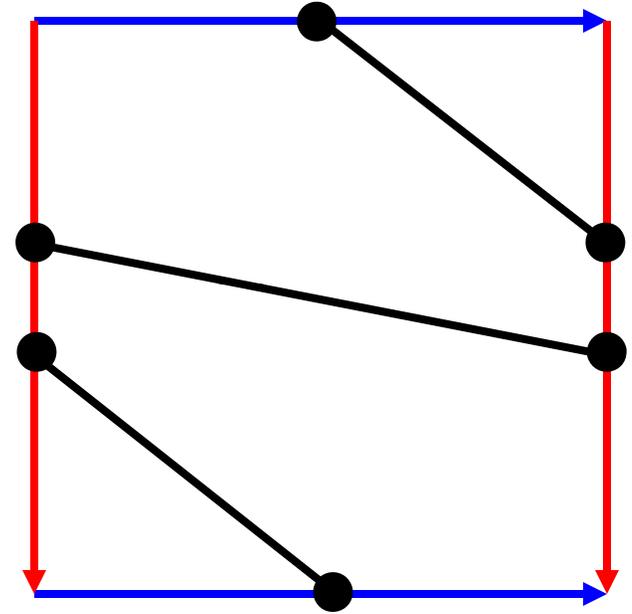
$$i(\alpha, \beta) = 2$$



EXAMPLE: Geometric intersection numbers are well understood on the torus



$(3,5)$



$(2,-1)$

$$\det \begin{array}{|c|c|} \hline 3 & 5 \\ \hline 2 & -1 \\ \hline \end{array} = -13$$

outline

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TOOL: Word equations.

What is a Dehn twist and why is it interesting?

Computing Dehn twists.

Open questions.

Algorithmic problems - History

Contractibility (Dehn 1912)

can shrink curve to point?

Transformability (Dehn 1912)

are two curves homotopy equivalent?

Schipper '92; Dey '94; Schipper, Dey '95
Dey-Guha '99 (linear-time algorithm)

Simple representative (Poincaré 1895)

can avoid self-intersections?

Reinhart '62; Ziechang '65; Chillingworth '69
Birman, Series '84

Algorithmic problems - History

Geometric intersection number

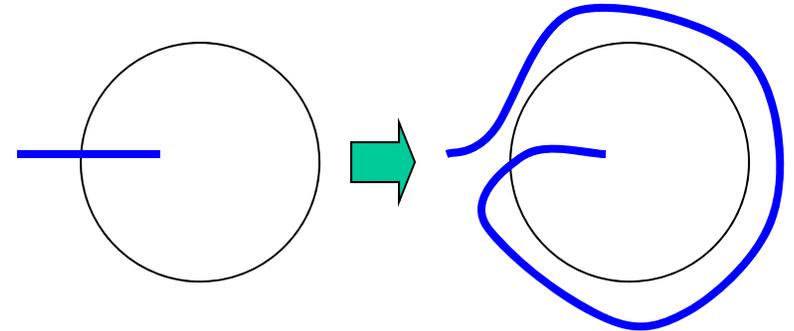
minimal number of intersections of two curves

Reinhart '62; Cohen, Lustig '87; Lustig '87;
Hamidi-Tehrani '97

polynomial only in explicit representations

Computing Dehn-twists

“wrap” curve along curve



Penner '84; Hamidi-Tehrani, Chen '96;
Hamidi-Tehrani '01

polynomial in compressed representations, but
only for fixed set of curves

Algorithmic problems – our results

Geometric intersection number

minimal number of intersections of two curves

Reinhart '62; Cohen, Lustig '87; Lustig '87;
Hamidi-Tehrani '97, [Schaefer-Sedgewick-Š '08](#)

polynomial in ~~explicit~~ [compressed](#) representations

Computing Dehn-twists

“wrap” curve along curve

Penner '84; Hamidi-Tehrani, Chen '96;
Hamidi-Tehrani '01, [Schaefer-Sedgewick-Š '08](#)

polynomial in compressed representations, for
~~fixed set of curves~~ [any pair](#) of curves

outline

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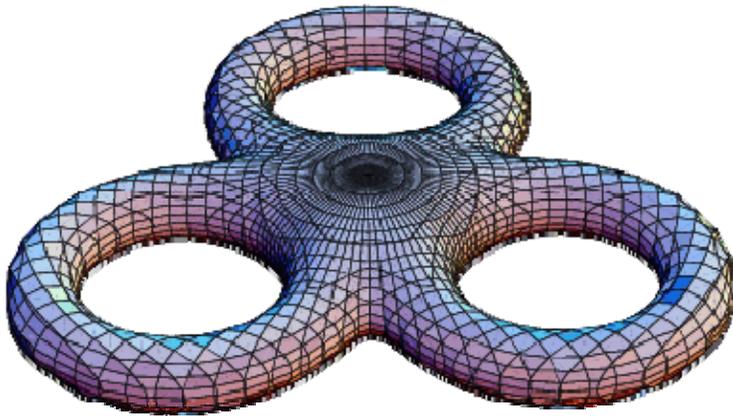
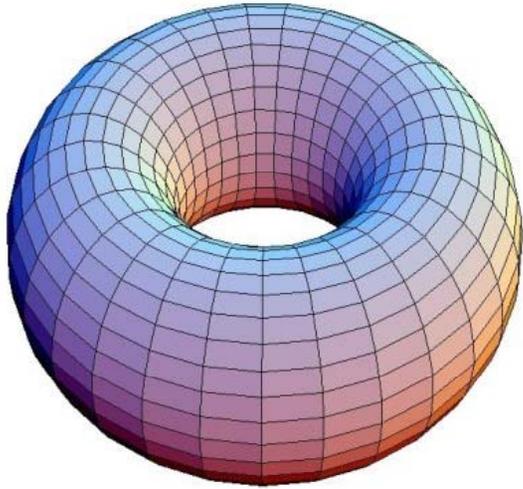
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Computing Dehn twists.

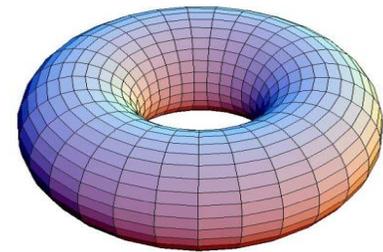
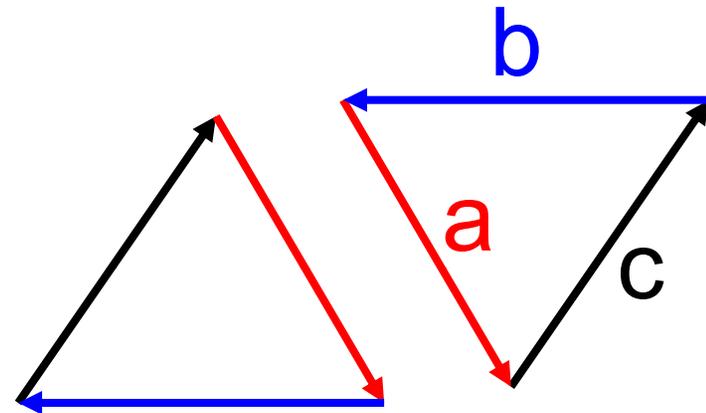
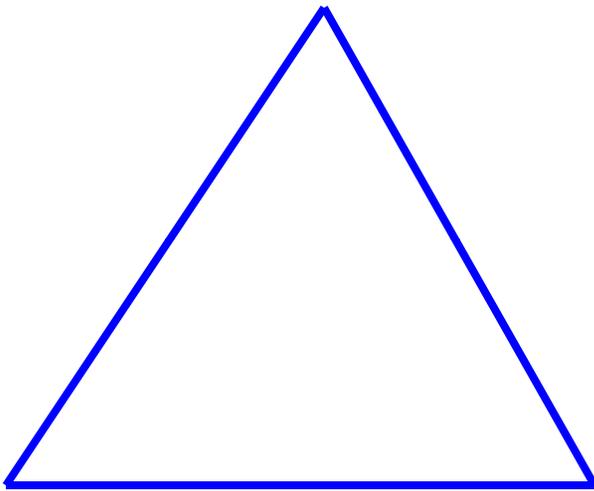
Open questions.

How to represent surfaces?



Combinatorial description of a surface

1. (pseudo) triangulation



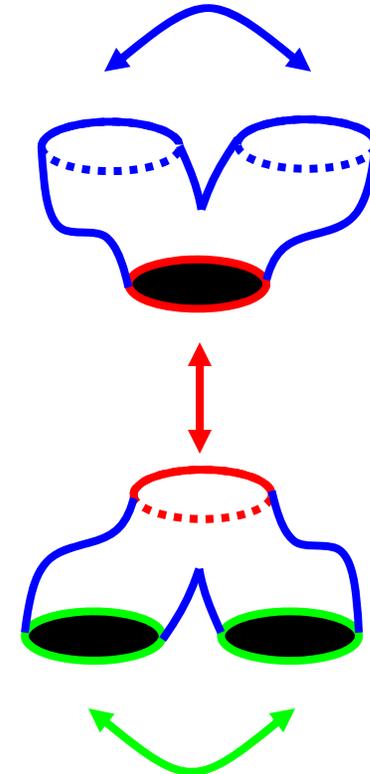
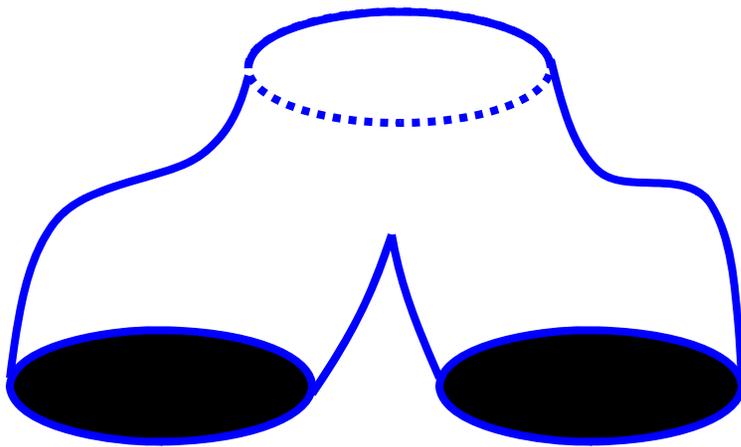
bunch of triangles

+ description of how to glue them

Combinatorial description of a surface



2. pair-of-pants decomposition



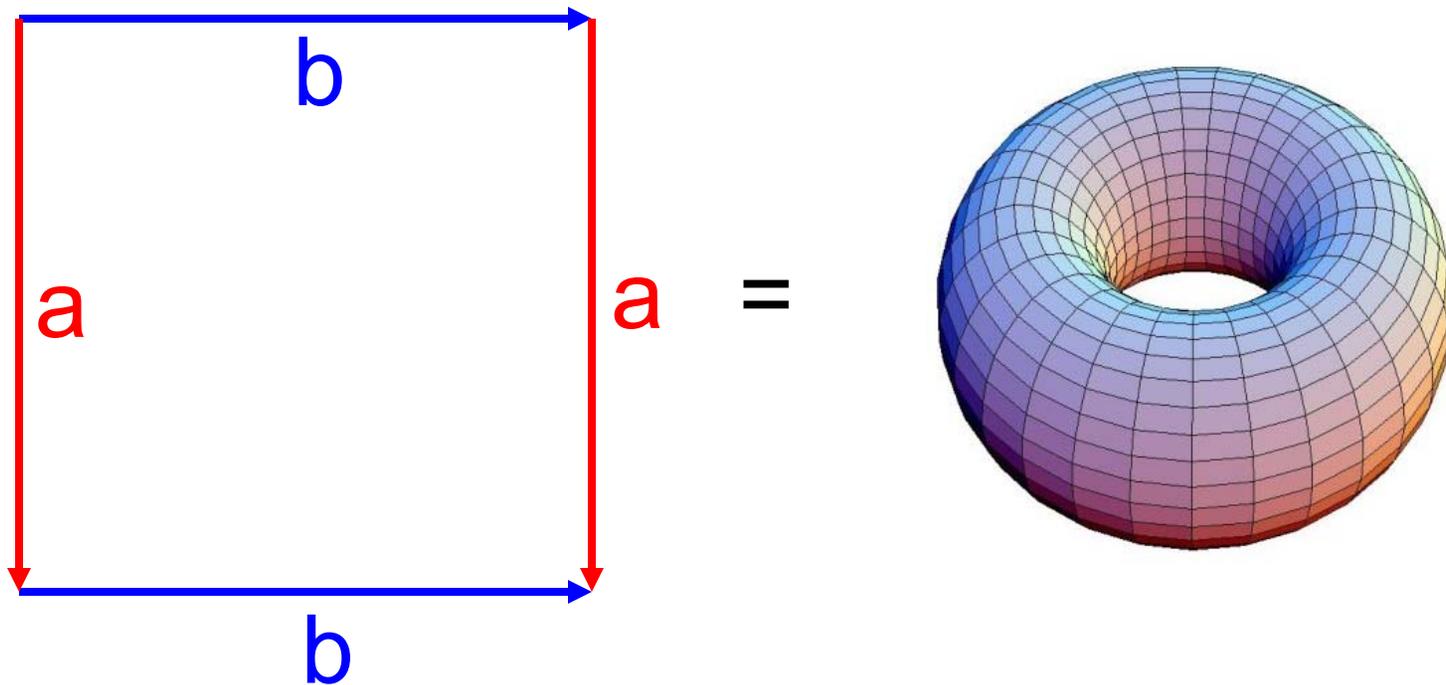
bunch of pair-of-pants

+ description of how to glue them

(cannot be used to represent: ball with ≤ 2 holes, torus)

Combinatorial description of a surface

3. polygonal schema



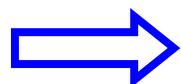
$2n$ -gon + pairing of the edges

outline

What is a curve?

Some history of algorithmic problems.

Representing surfaces.



Representing simple curves in surfaces.

Transforming between various representations.

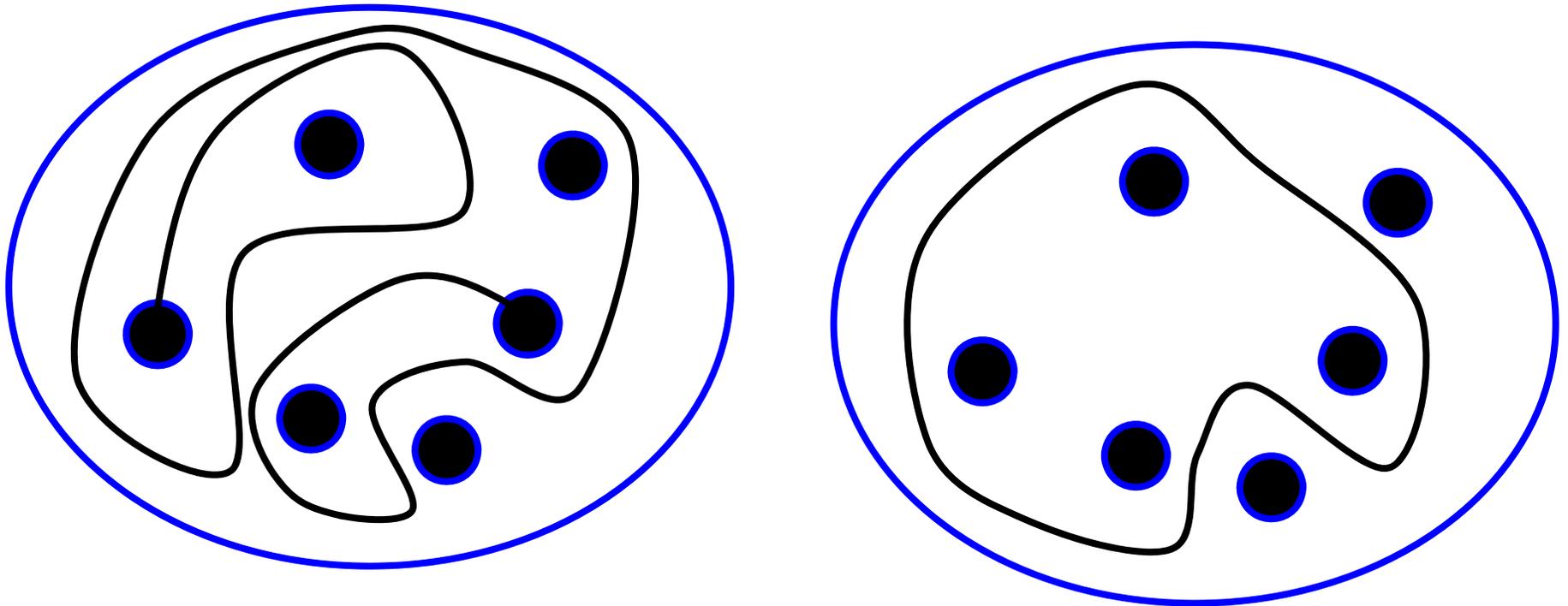
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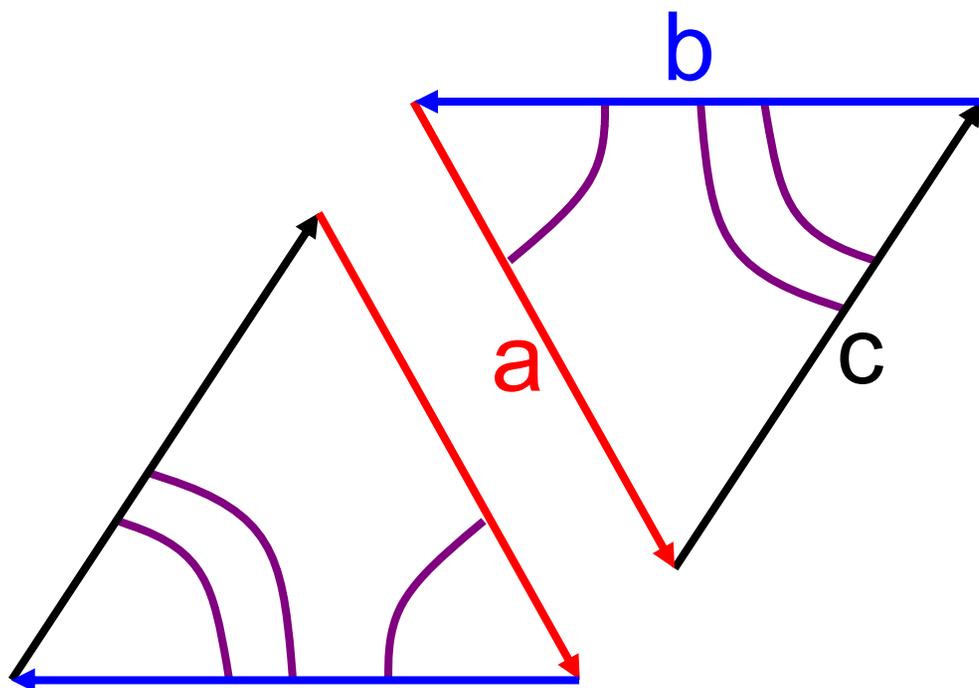
How to represent simple curves in surfaces (up to homotopy)?



Ideally the representation is “unique”
(each curve has a unique representation)

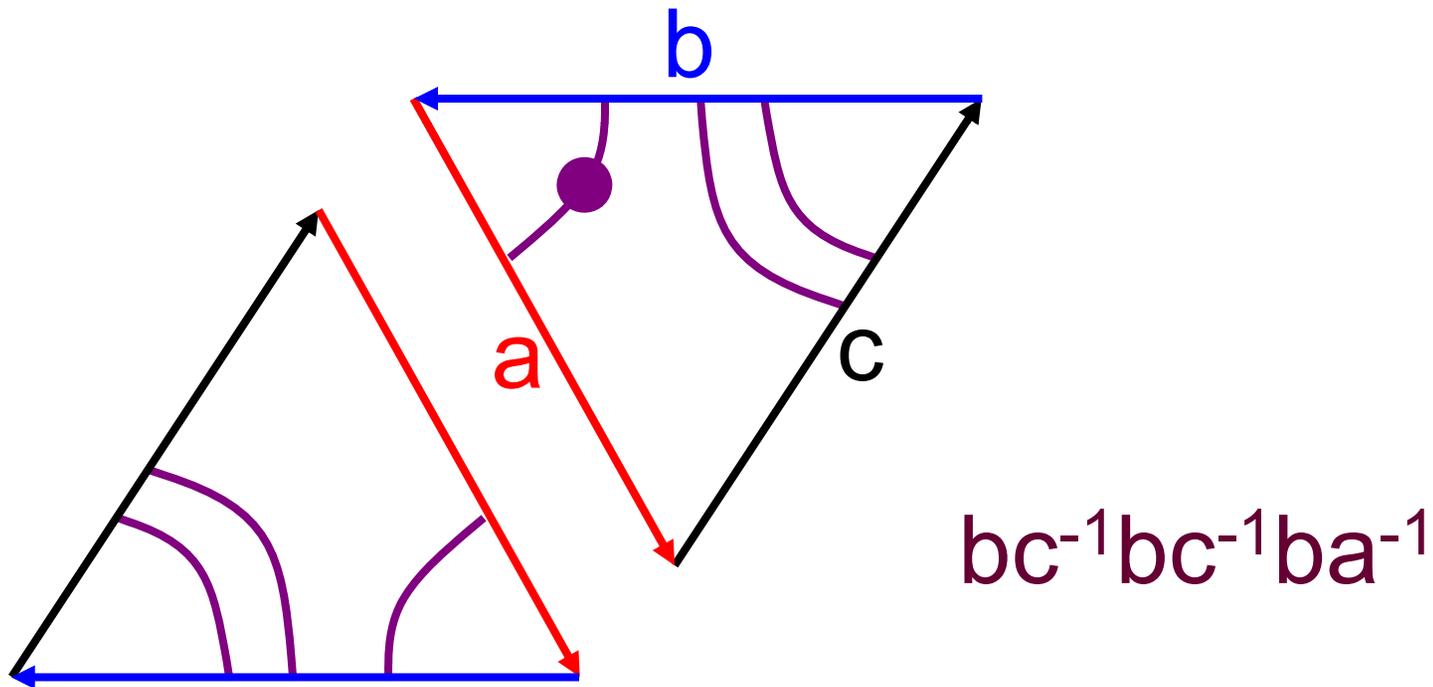
Combinatorial description of a (homotopy type of) a simple curve in a surface

1. intersection sequence with a triangulation



Combinatorial description of a (homotopy type of) a simple curve in a surface

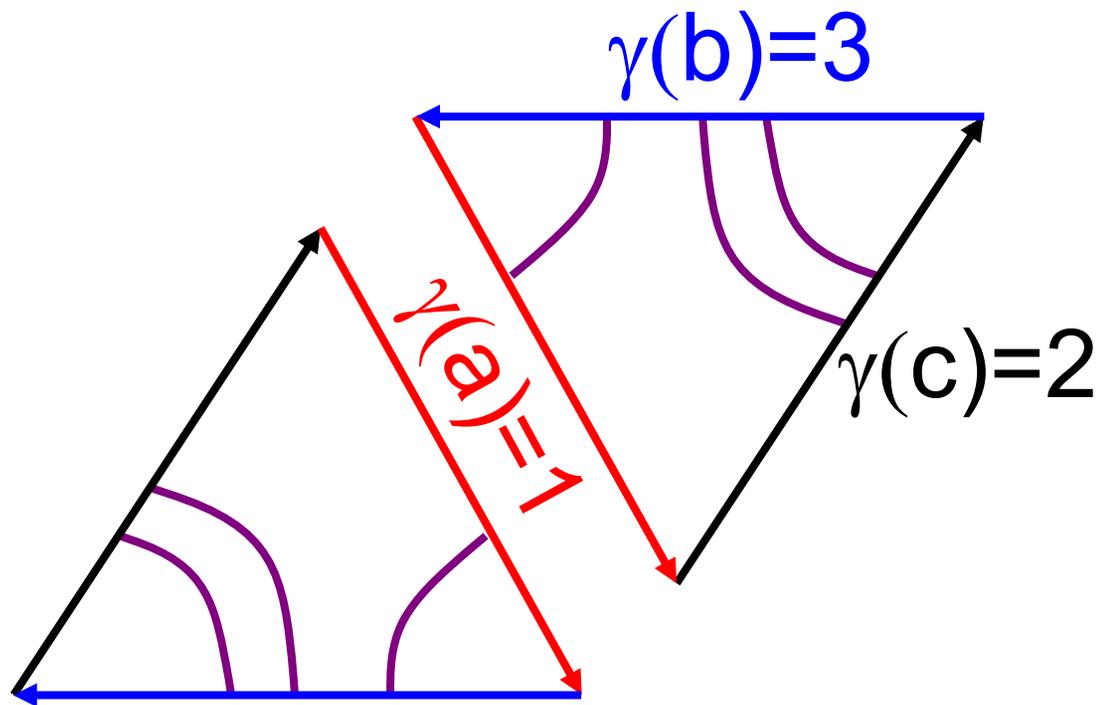
1. intersection sequence with a triangulation



almost unique if triangulation points on ∂S

Combinatorial description of a (homotopy type of) a simple curve in a surface

2. normal coordinates (w.r.t. a triangulation)

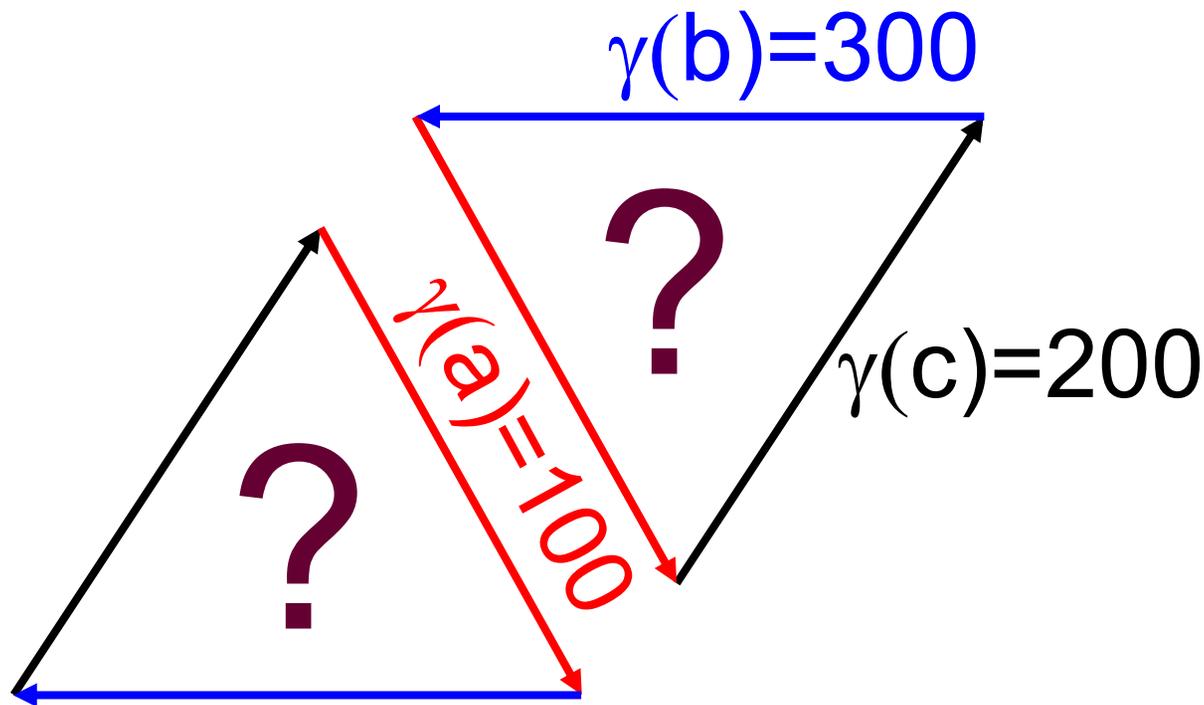


(Kneser '29)

unique if triangulation points on ∂S

Combinatorial description of a (homotopy type of) a simple curve in a surface

2. normal coordinates (w.r.t. a triangulation)



a very concise representation!

Recap:

1) how to represent them?

1. intersection sequence with a triangulation

$$bc^{-1}bc^{-1}ba^{-1}$$

2. normal coordinates (w.r.t. a triangulation)

$$\gamma(a)=1 \quad \gamma(b)=3 \quad \gamma(c)=2$$

2) what/how to compute?

geometric intersection number

outline

What is a curve?

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TOOL: Word equations.

What is a Dehn twist and why is it interesting?

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Open questions.

STEP1: Moving between the representations

1. intersection sequence with a triangulation

$$bc^{-1}bc^{-1}ba^{-1}$$

2. normal coordinates (w.r.t. a triangulation)

$$\gamma(a)=1 \quad \gamma(b)=3 \quad \gamma(c)=2$$

Can we move between these two representations efficiently?

$$\gamma(a)=1+2^{100} \quad \gamma(b)=1+3 \cdot 2^{100} \quad \gamma(c)=2^{101}$$

STEP1: Moving between the representations

1. intersection sequence with a triangulation

$$bc^{-1}bc^{-1}ba^{-1}$$

2. normal coordinates (w.r.t. a triangulation)

$$\gamma(a)=1 \quad \gamma(b)=3 \quad \gamma(c)=2$$

Can we move between these two representations efficiently?

YES

$$\gamma(a)=1+2^{100} \quad \gamma(b)=1+3 \cdot 2^{100} \quad \gamma(c)=2^{101}$$

Theorem (SSS'08):

normal coordinates \rightarrow compressed intersection sequence
in time $O(\sum \log \gamma(e))$

compressed intersection sequence \rightarrow normal coordinates
in time $O(|T|. \text{SLP-length}(S))$

compressed = straight line program (SLP)

$X_0 := a$

$X_1 := b$

$X_2 := X_1 X_1$

$X_3 := X_0 X_2$

$X_4 := X_2 X_1$

$X_5 := X_4 X_3$

$X_5 = bbbabb$

outline

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TOOL: Word equations.

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Open questions.

Main tool: Word equations

$$xax = yxy$$

x, y – variables

a, b - constants

Main tool: Word equations

$$xax = yxy$$

x, y – variables
 a, b - constants

a solution:

$$x = ab$$

$$y = ab$$

Word equations with given lengths

$$xayxb = axbxy$$

x, y – variables

a, b - constants

additional constraints: $|x|=4, |y|=1$

Word equations with given lengths

$$xayxb = axbxy$$

x, y – variables
 a, b – constants

additional constraints: $|x|=4, |y|=1$

a solution:

$$x=aaaa$$

$$y=b$$

Word equations

word equations

word equations with given lengths

Word equations

word equations - **NP-hard**

decidability – Makanin 1977

PSPACE – Plandowski 1999

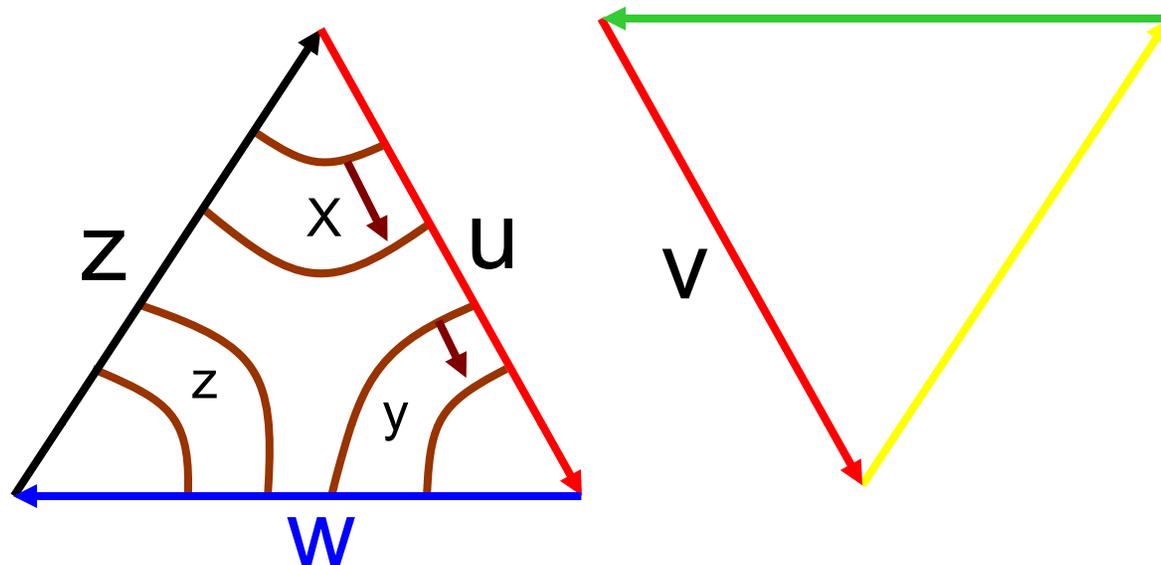
word equations with given lengths

Plandowski, Rytter '98 – **polynomial time** algorithm

Diekert, Robson '98 – **linear time** for quadratic eqns

(quadratic = each variable occurs ≤ 2 times)

Simulating curve using quadratic word equations



$$|u|=|v|=\gamma(u) \quad |x|=(|z|+|u|-|w|)/2$$

...

$$u=xy$$

...

$$v=u$$

—————→
Diekert-Robson

number of
components

Moving between the representations

1. intersection sequence with a triangulation

$$bc^{-1}bc^{-1}ba^{-1}$$

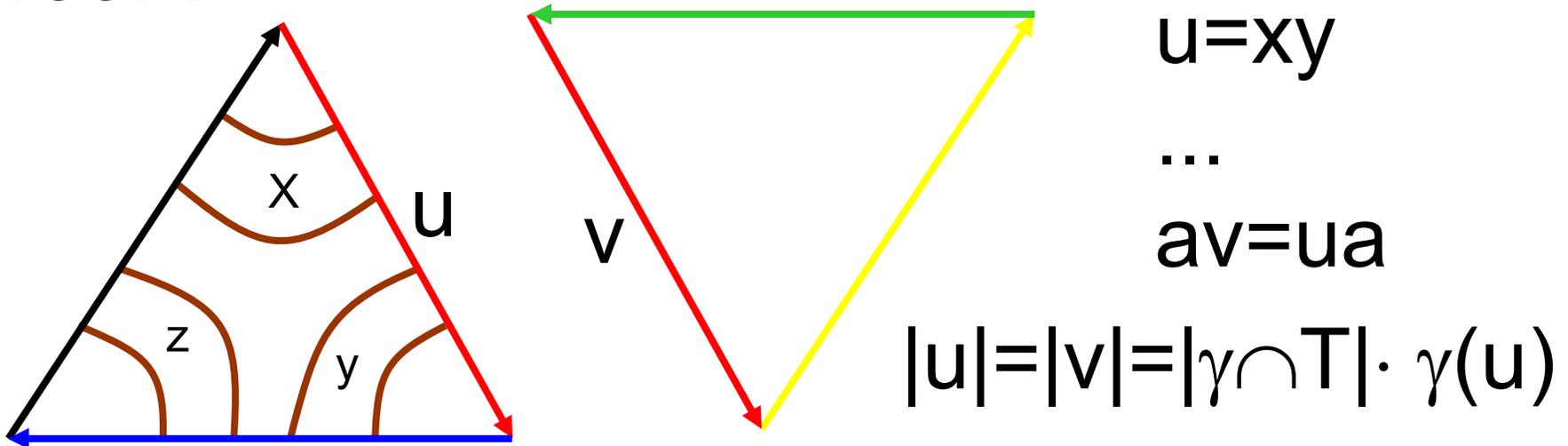
2. normal coordinates (w.r.t. a triangulation)

$$\gamma(a)=1 \quad \gamma(b)=3 \quad \gamma(c)=2$$

Theorem:

normal coordinates \rightarrow compressed intersection sequence
in time $O(\sum \log \gamma(e))$

“Proof”:



outline

What is a curve?

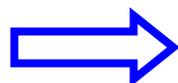
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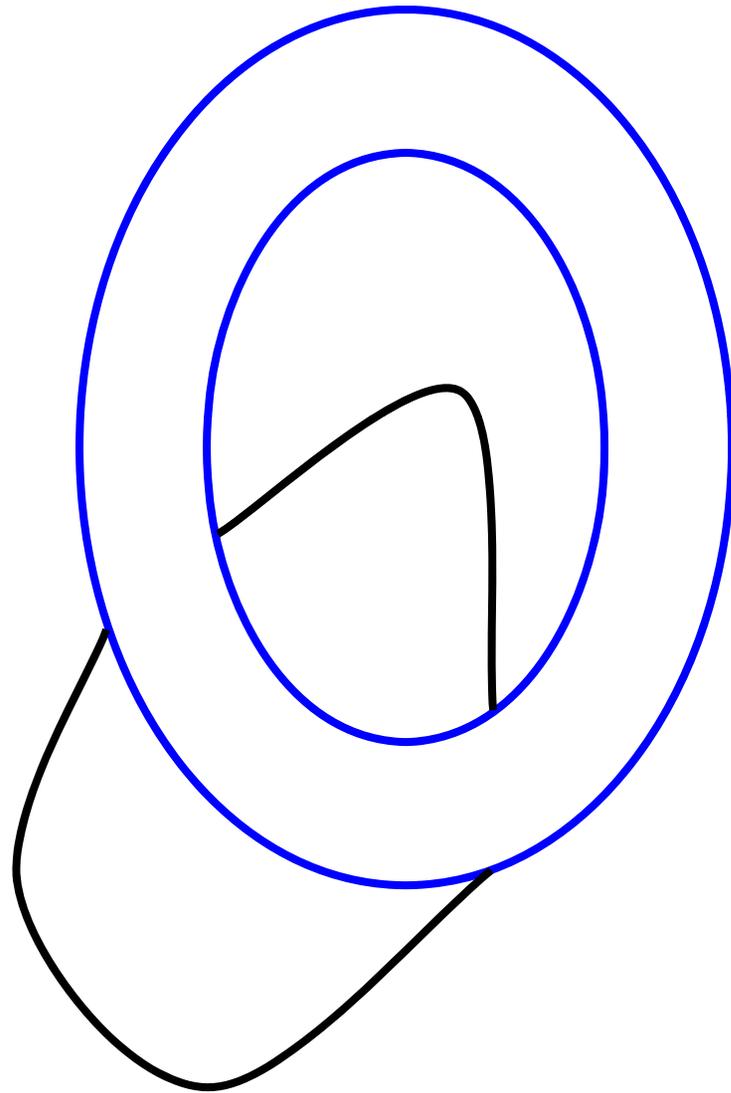
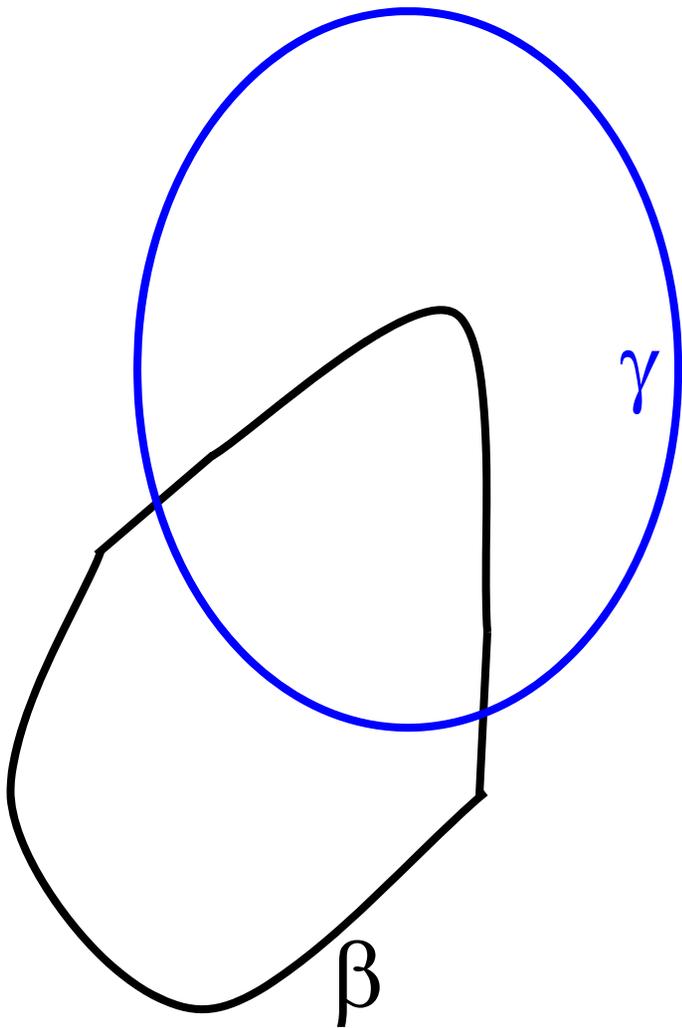


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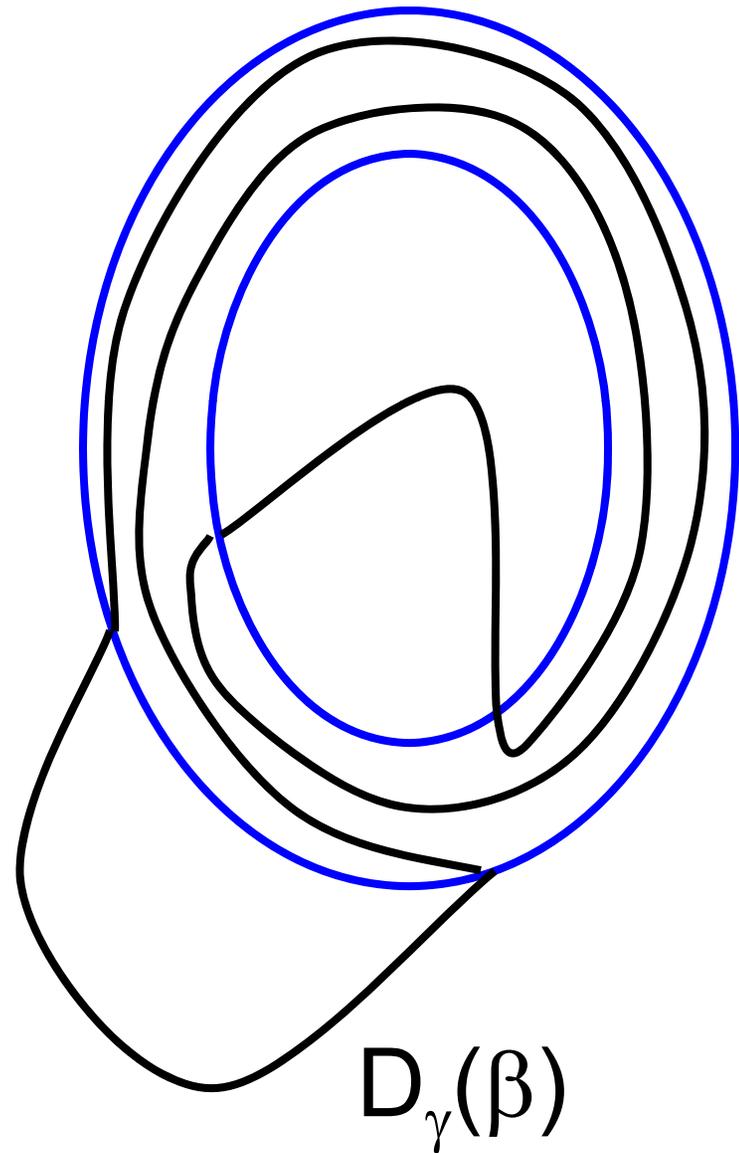
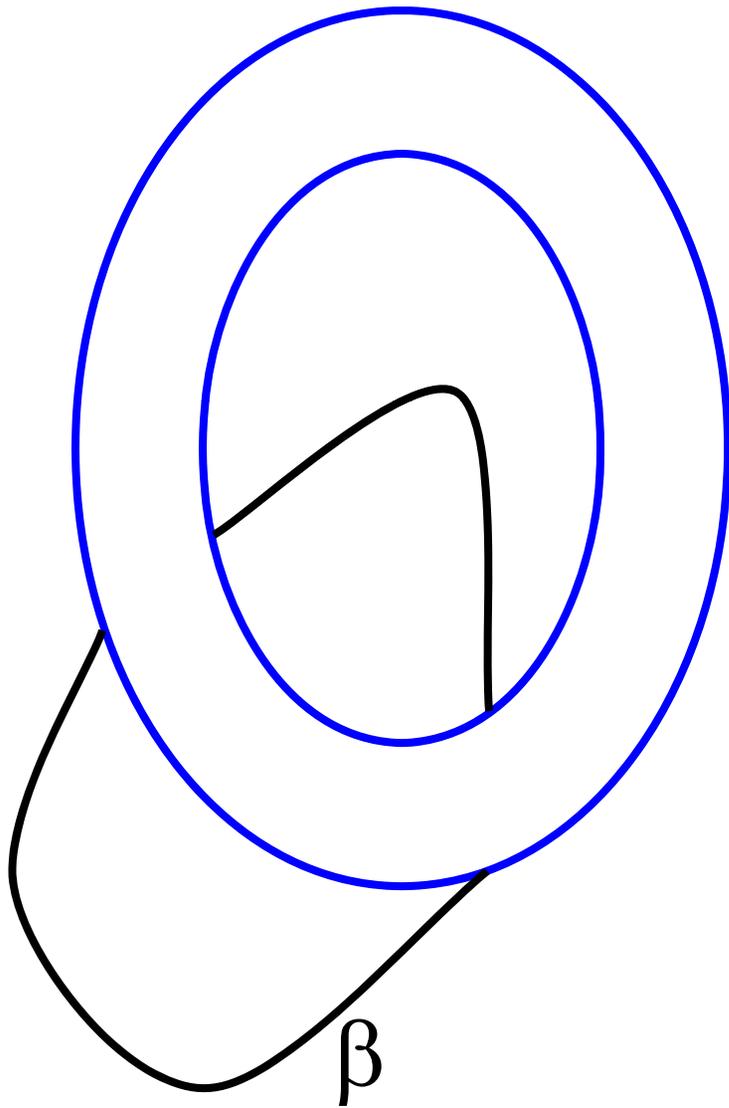
Computing Dehn twists.

Open questions.

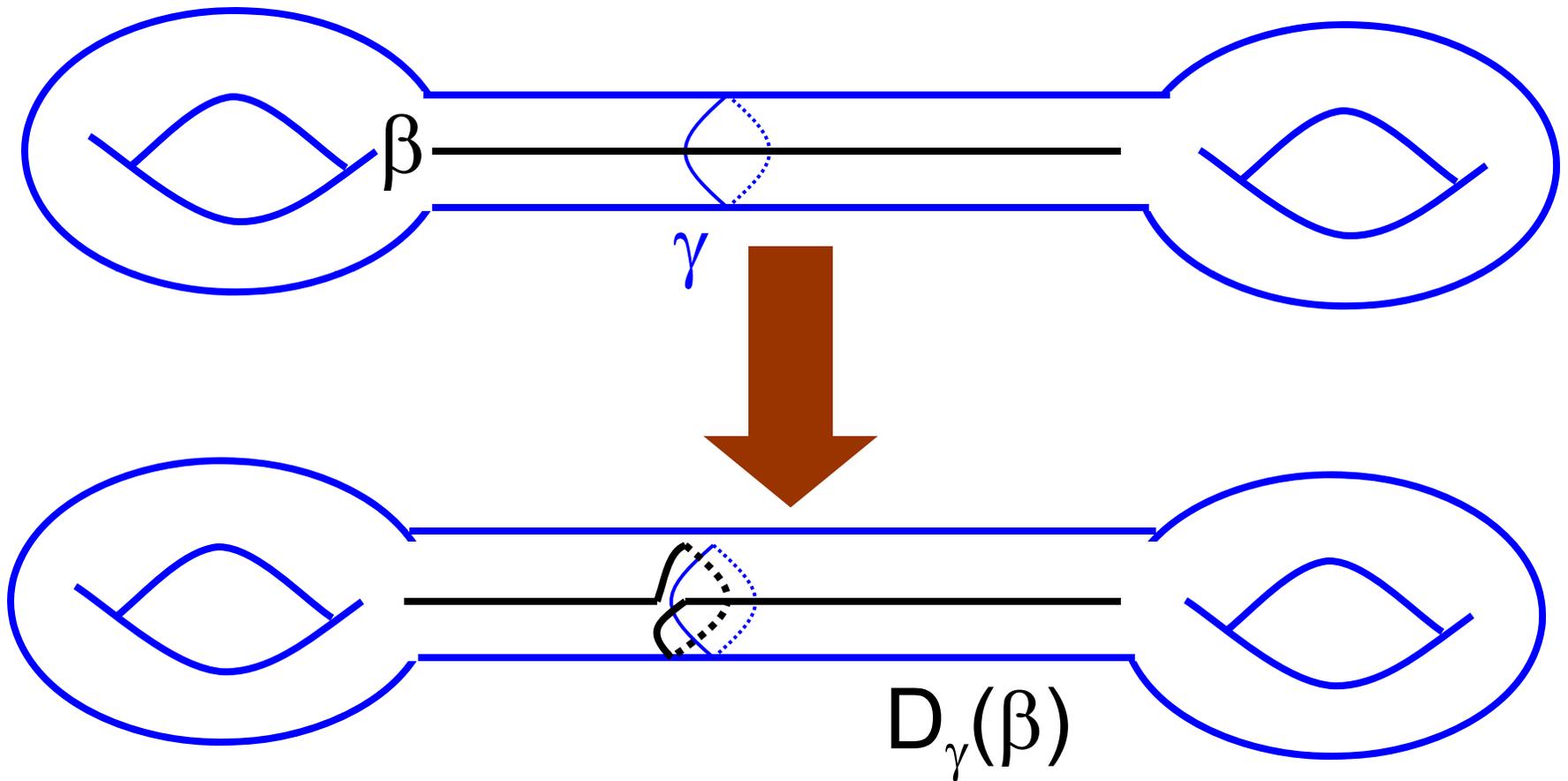
Dehn twist of β along γ



Dehn twist of β along γ

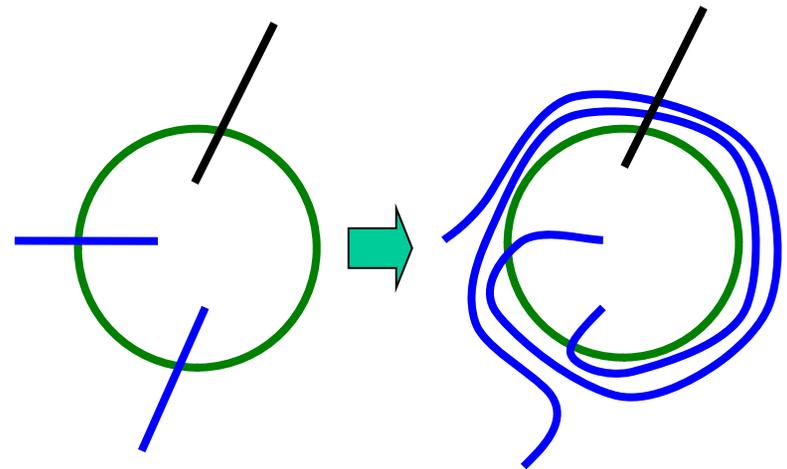
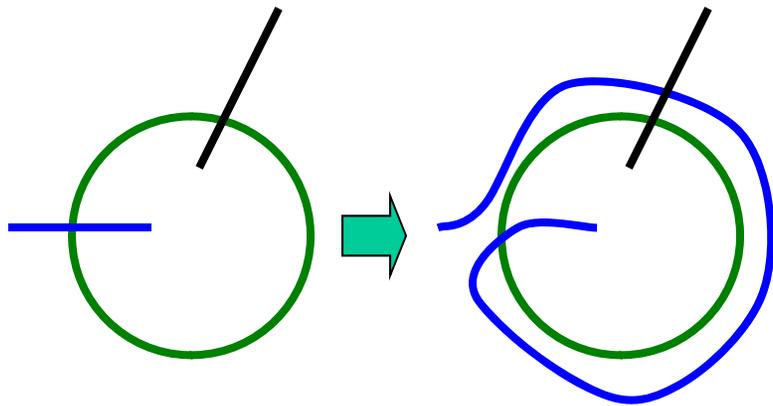


Dehn twist of β along γ



Geometric intersection numbers

$$i(\alpha, D_\gamma^n(\beta)) / i(\alpha, \gamma) \neq i(\gamma, \beta)$$



$$n \neq i(\alpha, \gamma) i(\gamma, \beta) - i(\alpha, \beta)$$

$$\leq i(\alpha, D_\gamma^n(\beta)) \leq$$

$$n \neq i(\alpha, \gamma) i(\gamma, \beta) + i(\alpha, \beta)$$

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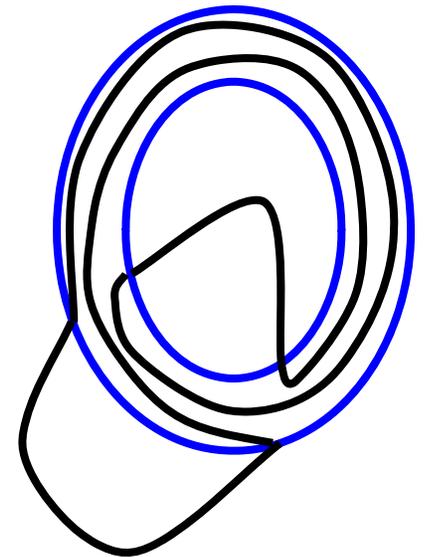


Computing Dehn twists.

Open questions.

Computing Dehn-Twists (outline)

1. normal coordinates ! word equations
with given lengths
2. solution = compressed intersection
sequence with triangulation
3. sequences ! (non-reduced) word for
Dehn-twist (substitution in SLPs)
4. Reduce the word ! normal coordinates



outline

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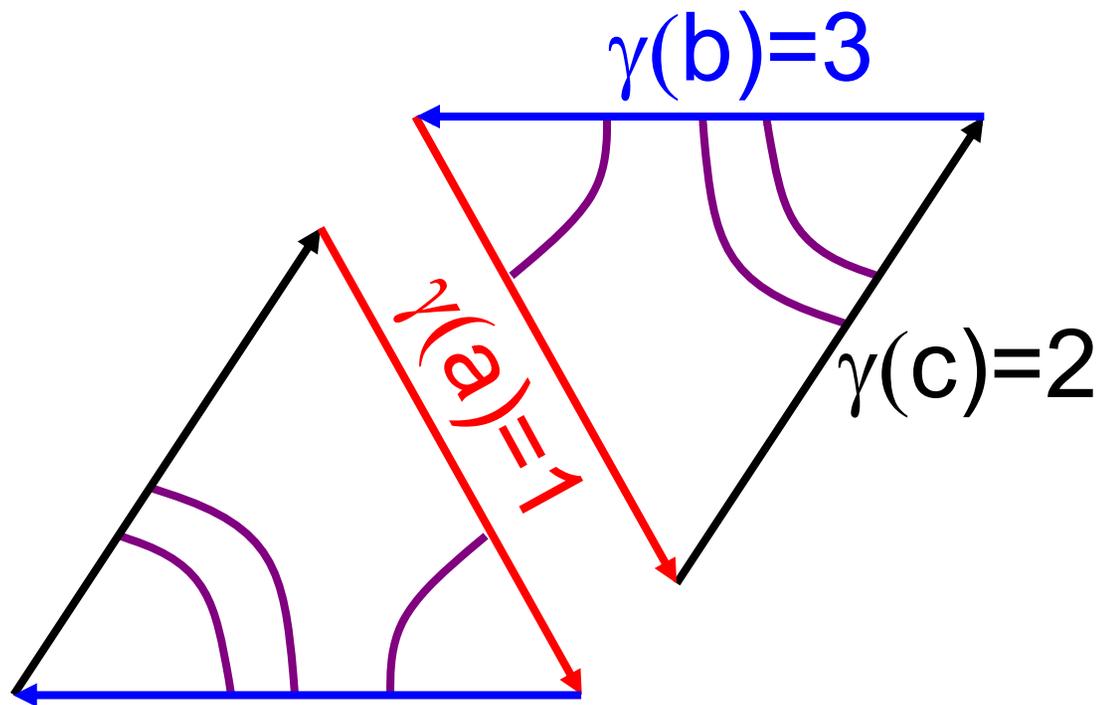
Computing Dehn twists.



Open questions.

PROBLEM #1: Minimal weight representative

2. normal coordinates (w.r.t. a triangulation)



unique if triangulation points on ∂S

PROBLEM #1: Minimal weight representative

INPUT: triangulation + gluing
normal coordinates of γ
edge weights

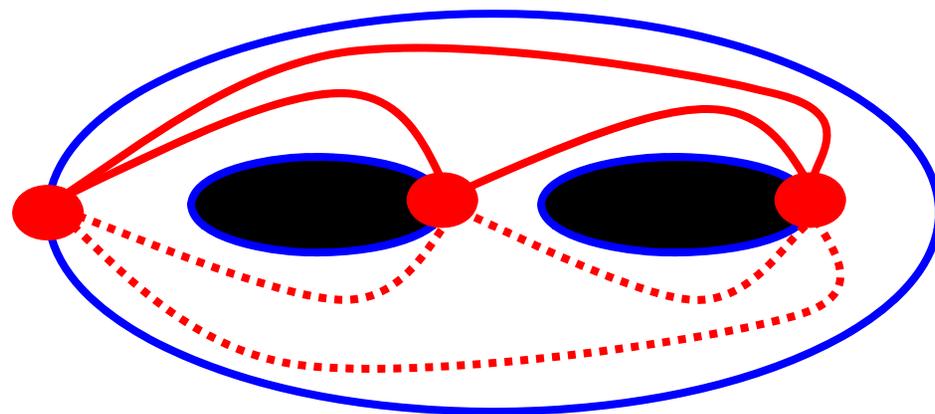
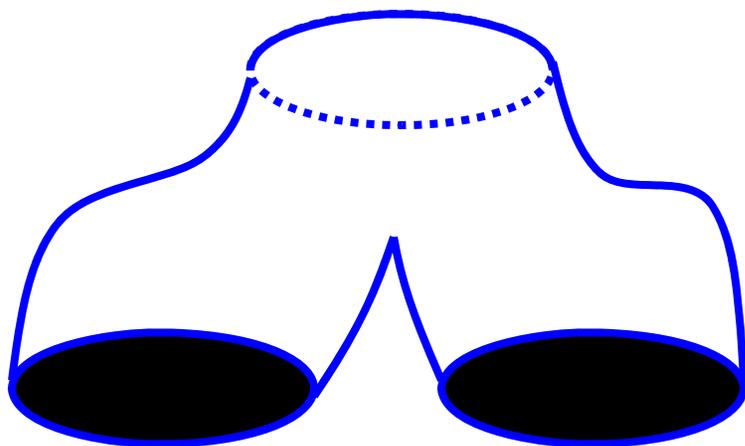
OUTPUT: $\gamma' \sim \gamma$

minimizing $\sum_{e \in T} \gamma'(e)$

PROBLEM #2: Moving between representations

3. Dehn-Thurston coordinates

(Dehn '38, W.Thurston '76)



unique representation for closed surfaces!

PROBLEM

normal coordinates \rightarrow Dehn-Thurston coordinates

in polynomial time? linear time?

PROBLEM #3: Word equations

NP-hard

decidability – Makanin 1977

PSPACE – Plandowski 1999

PROBLEM:

are word equations in NP?

are quadratic word equations in NP?

PROBLEM #4: Computing Dehn-Twists faster?

1. normal coordinates ! word equations
with given lengths

2. solution = compressed intersection
sequence with triangulation

3. sequences ! (non-reduced) word for
Dehn-twist (substitution in SLPs)

4. Reduce the word ! normal coordinates

$O(n^3)$ randomized, $O(n^9)$ deterministic

